Manifolds

Introduction

Composition of mappings. Let $f: A \to B, g: C \to D$. Define

$$\begin{aligned} g \circ f : f^{-1}(C) \to D, \\ g \circ f(a) = g(f(a)). \end{aligned}$$

Diffeomorphism. $f: U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ is called a diffeomorphism if U, V are open, f is bijective and $f, f^{-1} \in C^\infty$.

Local coordinate. Let $\mathcal{M} \neq \varnothing, U$ open in $\mathbb{R}^n$, $\varphi: U \to \mathcal{M}$ injective. Then φ is called a local coordinate on $\mathcal{M}$. If φ is also onto, then we call φ a global coordinate. Let ψ be another local coordinate on $\mathcal{M}$. Then φ and ψ are compatible if $\varphi^{-1} \circ \psi$ is a diffeomorphism.

Definition: Manifold. Let $\mathcal{M} \neq \varnothing, n \in \mathbb{N}^*, \varphi_\lambda : U_\lambda \subset \mathbb{R}^n \to \mathcal{M}$ are local coordinates on $\mathcal{M}, \lambda \in \Lambda$, satisfying (i) $\mathcal{M} = \bigcup_{\lambda \in \Lambda} \mathrm{Rg}\varphi_\lambda$, (ii) $\forall \lambda, \mu \in \Lambda, \varphi_\lambda$ and φ_μ are compatible.
We say $\mathcal{D} = \{\varphi_\lambda : \lambda \in \Lambda\}$ is a diffeomorphic structure on $\mathcal{M}$ and $(\mathcal{M}, \mathcal{D})$ is a n-dimensional smooth manifold.

Topology on a manifold. $\Omega \subset \mathcal{M}$ is open if $\forall \varphi \in \mathcal{D}, \varphi^{-1}(\Omega)$ open in $\mathbb{R}^n$. Then $\mathcal{T} = \{\Omega \text{ open in } \mathcal{M}\}$ is a topology on $\mathcal{M}$.

Assumption. (i) $\mathcal{T}$ is a Hausdorff topology on $\mathcal{M}$,
(ii) $\exists \varphi_k \in \mathcal{D}, k = 1, 2, \cdots$ such that $\mathcal{M} = \bigcup_{k=1}^{\infty} \mathrm{Rg}\varphi_k$.

Smooth maps on manifolds. Suppose $(\mathcal{M}, \mathcal{D}_{\mathcal{M}}), (\mathcal{N}, \mathcal{D}_{\mathcal{N}})$ are manifolds, $f : \mathcal{M} \to \mathcal{N}$ is continuous. For $\varphi \in \mathcal{D}_{\mathcal{M}}, \psi \in \mathcal{D}_{\mathcal{N}}, \psi^{-1} \circ f \circ \varphi$ is a coordinate expression of f.
f is smooth if every coordinate expresion of f is smoth. In this case, we write $f \in C^{\infty}(\mathcal{M}; \mathcal{N})$.

Diffeomorphism on manifolds. $f : \mathcal{M} \to \mathcal{N}$ is called a diffeomorphism if f is bijective and f, f^{-1} are smooth. We say $\mathcal{M}$ and $\mathcal{N}$ are diffeomorphic.

Homeomorphism $\not\to$ diffeomorphism: Milnor's 7-dimensional sphere.

Maximal diffeomorphic structure. Suppose $(\mathcal{M}, \mathcal{D})$ be a n-dim manifold. Write

$$\bar{\mathcal{D}} = \{\varphi \ n\text{-dim local coordinate on } \mathcal{M} : \varphi \text{ compatible with each } \psi \in \mathcal{D}\}.$$

Then $\bar{\mathcal{D}}$ is also a diffeomorphic structure on $\mathcal{M}$ and $(\mathcal{M}, \mathcal{D}), (\mathcal{M}, \bar{\mathcal{D}})$ are diffeomorphic.

Open submanifold. Let $(\mathcal{M}, \mathcal{D})$ be a manifold, $\Omega \neq \varnothing$ open in $\mathcal{M}$. Write

$$\mathcal{D}_{\Omega} = \left\{ \varphi\big|_{\varphi^{-1}(\Omega)} : \varphi \in \mathcal{D} \right\}.$$

Then $(\Omega, \mathcal{D}_{\Omega})$ is a manifold. We call Ω is an open submanifold of $\mathcal{M}$.

Local smoothness. Suppose $\mathcal{M}, \mathcal{N}$ are manifolds, $f : \mathcal{M} \to \mathcal{N}$.
Let $\Omega \subset \mathcal{M}$ open. We say f is smooth in Ω if $f\big|_{\Omega} : \Omega \to \mathcal{N}$ is smooth.
Let $p \in \mathcal{M}$. We say f is smooth at p if $\exists$ a neighborhood U of p such that f smooth in U.

Theorem: Partition of Unity. Let $\mathcal{M}$ be a connected manifold, $\mathcal{O}$ an open cover of $\mathcal{M}$.
Then $\exists \varphi_\lambda \in C_0^{\infty}(\mathcal{M}; [0, 1]), \lambda \in \Lambda$ such that
(i) $\{\varphi_\lambda : \lambda \in \Lambda\}$ is local finite, i.e. $\forall x \in \mathcal{M}, \exists$ neighborhood V of x and $\lambda_1, \cdots, \lambda_N \in \Lambda$ such that $\mathrm{supp}\varphi_\lambda \cap V = \varnothing, \forall \lambda \neq \lambda_i, i = 1, \cdots, N$.
(ii) $\forall \lambda \in \Lambda, \exists V \in \mathcal{O}, \mathrm{supp}\varphi_\lambda \subset V$.
(iii) $\sum_{\lambda \in \Lambda} \varphi_\lambda \equiv 1$ on $\mathcal{M}$.

Corollary: Existence of cut-off function. Let $p \in \mathcal{M}, U$ neighborhood of p. Then $\exists \varphi \in C_0^{\infty}(\mathcal{M}; [0, 1])$ such that (i) $\mathrm{supp}\varphi \subset U$, (ii) $\varphi \equiv 1$ near p.

Example: linear space. n-dim linear space $X = \mathrm{span}\{v_k\}_{k=1}^{n}$
Global coordinate: $\varphi : \mathbb{R}^n \to X, \varphi(x) = x^i v_i, x = (x^1, \cdots, x^n)$.

Example: graph of a smooth function. $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ smooth, Ω open. Define

$$\mathrm{graph} f = \{(x, f(x)) : x \in \Omega\} \subset \mathbb{R}^{m+n}.$$

Global coordinate: $\varphi : \Omega \to \mathrm{graph} f, x \mapsto (x, f(x))$.

Example: n-sphere. $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$.
Local coordinate: $\varphi_k^{\pm} : B_1 := \{x \in \mathbb{R}^n : |x| < 1\} \to S^n$, with

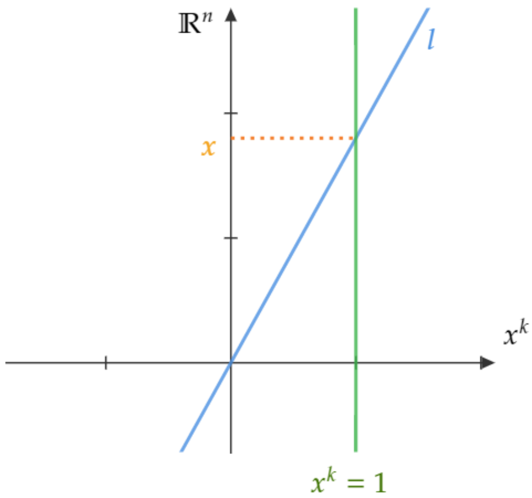
$$\varphi_k^{\pm}(x) = (x^1, \cdots, x^{k-1}, \pm\sqrt{1 - |x|^2}, x^{k+1}, \cdots, x^n),$$

where $k = 1, \cdots, n + 1$.

Example: projective space. $P_n = \{l \subset \mathbb{R}^{n+1} : l \text{ is a one-dim linear subspace}\}$. Local coordinate: $\varphi_k : \mathbb{R}^n \to P_n$ with

$$\varphi_k(x) = \mathbb{R}(x^1, \cdots, x^{k-1}, 1, x^k, \cdots, x^n),$$

where $k = 1, \cdots, n+1$.



Example: product manifold. $\mathcal{M}^m, \mathcal{N}^n$ are manifolds. For, $\varphi \in \mathcal{D}_\mathcal{M}, \psi \in \mathcal{D}_\mathcal{N}$, define

$$\varphi \times \psi : \mathrm{Dom}\varphi \times \mathrm{Dom}\psi \to \mathcal{M} \times \mathcal{N}, (x, y) \mapsto (\varphi(x), \psi(y))$$

and

$$\mathcal{D}_{\mathcal{M}\times\mathcal{N}} = \{\varphi \times \psi : \varphi \in \mathcal{D}_\mathcal{M}, \psi \in \mathcal{D}_\mathcal{M}\}.$$

Then $(\mathcal{M} \times \mathcal{N}, \mathcal{D}_{\mathcal{M}\times\mathcal{N}})$ is a $(m+n)$-dim manifold.

Tangent Space

Suppose $\mathcal{M}$ be a n-dim smooth manifold.

Linear mapping and linear functional. For linear spaces X and Y, denote the space of all linear mappings from X to Y by $\mathscr{L}(X; Y)$, and $\mathscr{L}(X) = \mathscr{L}(X; \mathbb{R})$.

Space C_p^∞. For $p \in \mathcal{M}$, define

$$C_p^\infty = \{f : \exists \text{ neighborhood } U \text{ of } p \text{ s.t. } f \in C^\infty(U)\}.$$

For $f, g \in C_p^\infty$, regard $f = g$ if $f \equiv g$ near p. For $\alpha, \beta \in \mathbb{R}, f, g \in C_p^\infty$, define

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x), (fg)(x) = f(x)g(x),$$

where $x \in \mathrm{Dom}f \cap \mathrm{Dom}g$. Then C_p^∞ is a linear space.

Smooth curve on manifold $\mathcal{M}$. $\gamma \in C^\infty((a, b); \mathcal{M}), -\infty \le a < b \le \infty$.

Definition: tangent. γ a smooth curve on $\mathcal{M}$ with $\gamma(t_0) = p$. Define $\gamma'(t_0) \in \mathscr{L}(C_p^\infty)$ by

$$\langle \gamma'(t_0), f \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} f \circ \gamma, \quad f \in C_p^\infty.$$

$\gamma'(t_0)$ is called the tangent of γ at t_0.

Definition: tangent space and tangent vector. The tangent space of $\mathcal{M}$ at p is

$$T_p\mathcal{M} = \{\gamma'(t_0) : \gamma \text{ curve on } \mathcal{M}, \gamma(t_0) = p\}.$$

$v \in T_p\mathcal{M}$ is called a tangent vector.
Remark. $p \neq q \implies T_p\mathcal{M} \cap T_q\mathcal{M} = \varnothing$.

Theorem. $T_p\mathcal{M}$ is a n-d linear space, where $n = \dim\mathcal{M}$. Given a local coordinate φ with $\varphi(x_0) = p$. Then a basis of $T_p\mathcal{M}$ is

$$\frac{\partial}{\partial x^i}\Big|_p = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi(x_0 + te_i).$$

Moreover, for a curve γ on $\mathcal{M}$ with $\gamma(t_0) = p$, write

$$\gamma^i = (\varphi^{-1} \circ \gamma)^i, \quad i = 1, \cdots, n.$$

Then $\gamma'(t_0) = \big(\gamma^i(t_0)\big)' \frac{\partial}{\partial x^i}\big|_p$.

Proof. Firstly we show that $\gamma'(t_0) = \big(\gamma^i(t_0)\big)' \frac{\partial}{\partial x^i}\big|_p$. Indeed, for $f \in C_p^\infty$,

$$\langle \frac{\partial}{\partial x^i}\Big|_p, f\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f \circ \varphi(x_0 + te_i) = \frac{\partial (f\circ\varphi)}{\partial x^i}(x_0).$$

Then

$$\begin{aligned}\langle \big(\gamma^i(t_0)\big)' \frac{\partial}{\partial x^i}\Big|_p, f\rangle &= \big(\gamma^i(t_0)\big)' \frac{\partial (f\circ\varphi)}{\partial x^i}(\gamma(t_0)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} f\circ\varphi(\gamma^1, \cdots, \gamma^n) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} f\circ\gamma = \langle \gamma'(t_0), f\rangle.\end{aligned}$$

Hence $\gamma'(t_0) = \big(\gamma^i(t_0)\big)' \frac{\partial}{\partial x^i}\big|_p$. From this indentity, we immediately get $T_p\mathcal{M}$ is a linear space spanned by $\{\frac{\partial}{\partial x^i}\big|_p\}_{i=1}^n$. Finally we show that $\{\frac{\partial}{\partial x^i}\big|_p\}_{i=1}^n$ are linearly independent. If for some $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$, $\alpha_i \frac{\partial}{\partial x^i}\big|_p = 0$, then

$$\langle \alpha^i \frac{\partial}{\partial x^i}\Big|_p, (\varphi^{-1})^j\rangle = \alpha^i \frac{(\varphi^{-1})^j \circ \varphi}{\partial x^i}(x_0) = \alpha^i \frac{\partial x^j}{\partial x^i}(x_0) = \alpha_j = 0.$$

Thus $\alpha_1 = \cdots = \alpha_n = 0$, and $\{\frac{\partial}{\partial x^i}\big|_p\}_{i=1}^n$ are linearly independent.

Propostion: properties of tangent vectors. Let $f, g \in C_p^\infty, v, w \in T_p\mathcal{M}, \alpha, \beta \in \mathbb{R}$.
(i) $f \equiv g$ near p\implies $\langle v, f\rangle = \langle v, g\rangle$.
(ii) $\langle v, \alpha f + \beta g\rangle = \alpha\langle v, f\rangle + \beta\langle v, g\rangle$.
(iii) $\langle \alpha v + \beta w, f\rangle = \alpha\langle v, f\rangle + \beta\langle w, f\rangle$.
(iv) $\langle v, fg\rangle = f(p)\langle v, g\rangle + g(p)\langle v, f\rangle$.

Drivetive on a linear space. Let X be a n-d linear space, Ω open in $X, p \in \Omega$. Define $L : X \to T_p\Omega$ by

$$L(v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (p + tv), \quad v \in X.$$

Then L is a linear isomorphism from X \to $T_p\Omega$. In this sense, we regard $T_p\Omega = X$ and $v = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(p + tv)$.

Proof. Let $E_1, \cdots, E_n$ be a basis of X, $\varphi(x) = x^i E_i$ with $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Then

$$\begin{aligned} L(v^i E_i) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (p + tv^i E_i) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left(\varphi^{-1}(p + tv^i E_i)\right)^j \frac{\partial}{\partial x^j}\Big|_p \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left(\varphi^{-1}(p) + tv^i e_i\right)^j \frac{\partial}{\partial x^j}\Big|_p \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left(\left(\varphi^{-1}(p)\right)^j + tv^j\right) \frac{\partial}{\partial x^j}\Big|_p \\ &= v^j \frac{\partial}{\partial x^j}\Big|_p. \end{aligned}$$

Differentiation

Definition: differentiation on manifolds. For $f \in C^\infty(\mathcal{M}; \mathcal{N})$ and $p \in \mathcal{M}$, define $df_p : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ by

$$df_p(v) = \frac{d}{dt}\Big|_{t=t_0} f \circ \gamma, \quad v \in T_p\mathcal{M},$$

where γ is a curve on $\mathcal{M}$ such that $\gamma(t_0) = p, \gamma'(t_0) = v$.

Coordinate expression of differentiation. Let $\varphi = \varphi(x) \in \mathcal{D}_\mathcal{M}$ with $p = \varphi(x_0)$, $\psi = \psi(y) \in \mathcal{D}_\mathcal{N}$ with $f(p) = \psi(y_0)$, $\tilde{f} = \psi^{-1} \circ f \circ \varphi$ and $\gamma^i = (\varphi^{-1} \circ \gamma)^i$. Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} f \circ \gamma =& \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} (\psi^{-1} \circ f \circ \gamma)^j \frac{\partial}{\partial y^j}\Big|_{f(p)} \\ =& \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \tilde{f}^j(\gamma^1, \cdots, \gamma^n) \frac{\partial}{\partial y^j}\Big|_{f(p)} \\ =& (\gamma^i)'(t_0) \frac{\partial \tilde{f}^j}{\partial x^i}(x_0) \frac{\partial}{\partial y^j}\Big|_{f(p)}. \end{aligned}$$

Proposition: properties of differentiation. Let $f, g \in C^\infty(\mathcal{M}; \mathcal{N}), v, w \in T_p\mathcal{M}, \alpha, \beta \in \mathbb{R}, h \in C^\infty(\mathcal{N}; \mathcal{P})$.
(i) $f \equiv g$ near $p \implies df_p = dg_p$.
(ii) $df_p(\alpha v + \beta w) = \alpha df_p(v) + \beta df_p(w)$.
(iii) $d(h \circ f)_p = dh_{f(p)} \circ df_p$.

Proof. (i)(ii) are obvious.
(iii) For curve γ on $\mathcal{M}$ with $\gamma(t_0) = p, \gamma'(t_0) = v$, we have

$$d(h \circ f)_p(v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} h \circ (f \circ \gamma) = dh_{f(p)}\left((f \circ \gamma)'(t_0)\right) = dh_{f(p)}(df_p(v)).$$

Useful formulas.
(i) Let $I_\mathcal{M}$ be the indentity map on $\mathcal{M}$. Then $d(I_\mathcal{M})_p = I_{T_p}\mathcal{M}, \forall p \in M$.
(ii) Let $f \in C^\infty(\mathcal{M})$. Then $df_p(v) = \langle v, f \rangle, \forall v \in T_p\mathcal{M}$.
(iii) Let γ be a curve on $\mathcal{M}$. Then $d\gamma_{t_0}(1) = \gamma'(t_0)$.

Proof. (i) $d(I_\mathcal{M})_p(\gamma'(t_0)) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} I_\mathcal{M} \circ \gamma = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} \gamma = \gamma'(t_0)$.
(ii) $df_p(\gamma'(t_0)) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} f \circ \gamma = \langle v, f \rangle$.
(iii) $d\gamma_{t_0}(1) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \gamma(t + t_0) = \gamma'(t_0)$.

Tangent Bundle

Definition: tangent bundle. Given a n-d manifold $(\mathcal{M}, \mathcal{D})$. Write

$$T\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M}.$$

Set

$$\hat{\mathcal{D}} = \{\hat{\varphi} : \varphi \in \mathcal{D}\}, \quad \hat{\varphi}(x, v) = v^i \frac{\partial}{\partial x^i}\Big|_{\varphi(x)}, x, v \in \mathbb{R}^n.$$

Then $(T\mathcal{M}, \hat{\mathcal{D}})$ is a $2n$-d manifold, which called the tangent bundle of $\mathcal{M}$.

Definition: smooth vector field. $X \in C^\infty(\mathcal{M}; T\mathcal{M})$ is called a smooth vetor field if

$$X(p) \in T_p\mathcal{M}, \quad \forall p \in \mathcal{M}.$$

Let $\varphi = \varphi(x)$ be a local coordinate on an open set Ω in $\mathcal{M}$. Write

$$\frac{\partial}{\partial x^i} : \Omega \to T_p\Omega, p \mapsto \frac{\partial}{\partial x^i}\Big|_p = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi\big(\varphi^{-1}(p) + te_i\big).$$

Then $\frac{\partial}{\partial x^i}$ is a smooth vector field on Ω. $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$ is a basis vetor field.

Proof. The coordinate expression of X is

$$\hat{\varphi}^{-1} \circ \frac{\partial}{\partial x^i} \circ \varphi(x) = (x, e_i) \in C^\infty.$$

Thus $\frac{\partial}{\partial x^i} \in C^\infty(\Omega; T_p\Omega)$.

Proposition: components of a vector field. Let $X : \mathcal{M} \to T, X(p) \in T_p\mathcal{M}$. Define $X^i : \Omega \to \mathbb{R}$ by

$$X(p) = X^i(p) \frac{\partial}{\partial x^i}\Big|_p, \quad p \in \Omega.$$

Then X is a vector field if and only if $X^i \in C^\infty(\Omega), i = 1, \cdots, n$.

Proof. Since

$$\hat{\varphi}^{-1} \circ X \circ \varphi(x) = \big(x, (X^i \circ \varphi)e_i\big) = (x, X^1 \circ \varphi, \cdots, X^n \circ \varphi),$$

X is a vector field $\iff$ each $X^i \circ \varphi$ smooth $\iff$ each X^i smooth.

Proposition. $T\Omega = \Omega \times \mathbb{R}^n$ (with coordinate $\varphi = \varphi(x)$).

Proof. Define a one-to-one mapping $\psi : T\Omega \to \Omega \times \mathbb{R}^n$ by

$$\psi\left(v^i \frac{\partial}{\partial x^i}\Big|_p\right) = (p, v^1, \cdots, v^n).$$

$\mathcal{V}(\mathcal{M}) = \{X : X \text{ smooth vector field on } \mathcal{M}\}.$

Product of a smooth function and a vector field. Let $f, g \in C^\infty(\mathcal{M}), X, Y \in \mathcal{V}(\mathcal{M})$. Define

$$(fX + gY)(p) = f(p)X_p + g(p)Y_p, \quad p \in M.$$

Then smoothness of components of X, Y implies $fX + gY \in \mathcal{V}(\mathcal{M})$. Hence $\mathcal{V}(\mathcal{M})$ is a C^∞-module.

Action of a vector field on a smooth function. For $X \in \mathcal{V}(\mathcal{M}), f \in C^\infty(\mathcal{M})$, define Xf by

$$(Xf)(p) = \langle X_p, f \rangle, \quad p \in \mathcal{M}.$$

Then $Xf \in C^\infty(\mathcal{M})$.
If $X = X^i \frac{\partial}{\partial x^i}$, then $Xf = X^i \frac{\partial f}{\partial x^i}$, where $\frac{\partial f}{\partial x^i} = \frac{\partial (f \circ \varphi)}{\partial x^i} \circ \varphi^{-1}$.

Product of two vector fields. Let $X, Y \in \mathcal{V}(\mathcal{M})$. Define $XY : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ by

$$(XY)(f) = X(Yf), \quad f \in C^\infty(\mathcal{M}).$$

Then XY is a linear differential operator of 2-order. The commutator of X, Y is $XY - YX$.

Proposition: Lie bracket. Let $X, Y \in \mathcal{V}(\mathcal{M})$. Then there exists a unique vector field $[X, Y] \in \mathcal{V}(\mathcal{M})$ such that

$$[X, Y]f = (XY - YX)f, \quad \forall f \in C^\infty(\mathcal{M}).$$

We call $[X, Y]$ the Lie bracket of X and Y.

Coordinate expression of Lie bracket. Let $X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i}$, then

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

Proof.

$$\begin{aligned}
[X, Y]f =& X^i \frac{\partial}{\partial x^i} \left(Y^j \frac{\partial f}{\partial x^j} \right) - Y^i \frac{\partial}{\partial x^i} \left(X^j \frac{\partial f}{\partial x^j} \right) \\
=& \left(X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} + X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} \right) \\
&- \left(Y^i \frac{\partial X^j}{\partial x^i} \frac{\partial f}{\partial x^j} + Y^i X^j \frac{\partial^2 f}{\partial x^i \partial x^j} \right) \\
=& \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial f}{\partial x^j}.
\end{aligned}$$

Proposition: properties of Lie bracket. Let $X, Y, Z \in \mathcal{V}(\mathcal{M}), \alpha, \beta \in \mathbb{R}$.
(i) $[X, Y] = -[Y, X]$.
(ii) $[\alpha X + \beta Y, Z] = \alpha [X, Z] + \beta [Y, Z]$.
(iii) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi's indentity).

Proof. (i)(ii) are obvious.
(iii) Write $X_{3i} = X, X_{3i+1} = Y, X_{3i+2} = Z, i \in \mathbb{Z}$. Then

$$
\begin{aligned}
&\sum_{k=0}^{2}[[X_k, X_{k+1}], X_{k+2}] \\
=&\sum_{k=0}^{2}[X_k X_{k+1} - X_{k+1}X_k, X_{k+2}] \\
=&\sum_{k=0}^{2}\left(X_k X_{k+1} X_{k+2} - X_{k+2}X_k X_{k+1} - X_{k+1}X_k X_{k+2} + X_{k+2}X_{k+1}X_k\right) \\
=&\sum_{k=0}^{2} X_k X_{k+1} X_{k+2} - \sum_{k=0}^{2} X_k X_{k-2} X_{k-1} - \sum_{k=0}^{2} X_{k+1} X_k X_{k+2} + \sum_{k=0}^{2} X_{k+1} X_k X_{k-1} \\
=&\sum_{k=0}^{2} X_k X_{k+1} X_{k+2} - \sum_{k=0}^{2} X_k X_{k+1} X_{k+2} - \sum_{k=0}^{2} X_{k+1} X_k X_{k+2} + \sum_{k=0}^{2} X_{k+1} X_k X_{k+2} \\
=&0.
\end{aligned}
$$

Definition: diffeomorphism. Let $F : \mathcal{M} \to \mathcal{N}$ be a diffeomorphism. Define $dF : \mathcal{V}(\mathcal{M}) \to \mathcal{V}(\mathcal{N})$ by

$$
dF(X)(F(p)) = dF_p(X(p)), \quad X \in \mathcal{V}(\mathcal{M}), p \in \mathcal{M}.
$$

$$
dF\left(X^i \frac{\partial}{\partial x^i}\right) = (X^i \circ F^{-1}) dF\left(\frac{\partial}{\partial x^i}\right).
$$

Proposition. $dF([X, Y]) = [dF(X), dF(Y)]$.

Proof. For $g \in C^\infty(\mathcal{M}; \mathcal{N})$,

$$
\begin{aligned}
\langle dF([X, Y]), g\rangle =& \langle [X, Y], g \circ F\rangle \\
=& \left\langle \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right) \frac{\partial}{\partial x^j}, g \circ F \right\rangle \\
=& \left\langle \left(\left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right) \circ F^{-1}\right) dF\left(\frac{\partial}{\partial x^j}\right), g \right\rangle.
\end{aligned}
$$

Since

$$
dF\left(\frac{\partial}{\partial x^i}\right)(Y^j \circ F^{-1}) = \frac{\partial Y^j}{\partial x^i} \circ F^{-1},
$$

we get $\langle dF([X, Y]), g\rangle = \langle [dF(X), dF(Y)], g\rangle$.

Riemannian Manifolds

Bilinear Forms

Definition: bilinear form. Let $B : \mathcal{V}(\mathcal{M}) \times \mathcal{V}(\mathcal{M}) \to C^\infty(\mathcal{M})$. If

$$
\begin{gathered}
B(fX + gY, Z) = fB(X, Z) + gB(Y, Z), \\
B(X, fY + gZ) = fB(X, Y) + g(X, Z)
\end{gathered}
$$

for any $f, g \in C^\infty(\mathcal{M}), X, Y, Z \in \mathcal{V}(\mathcal{M})$, then we call B a bilinear form.

Proposition. Let $F : \mathcal{V}(\mathcal{M}) \to C^\infty(\mathcal{M})$ satisfying

$$F(fX + gY) = fF(X) + gF(Y), \quad f, g \in C^\infty(\mathcal{M}), X, Y \in \mathcal{V}(\mathcal{M}).$$

For $p \in \mathcal{M}$, if $X_p = Y_p$, then $F(X)(p) = F(Y)(p)$.

Proof. Choose a cut-off function $\zeta \in C^\infty(\mathcal{M})$ such that $\zeta(p) = 1$ and $\mathrm{supp}\zeta \subset \mathrm{Rg}\varphi$, where $\varphi = \varphi(x)$ is a local coordinate near p. For $X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i}$ near p, $X_p = Y_p \implies X^i(p) = Y^i(p), \forall i$, which implies

$$\begin{aligned} F(X)(p) = F(\zeta^2 X)(p) = F\left(\zeta^2 X^i \frac{\partial}{\partial x^i}\right)(p) \\ = (\zeta X^i)(p) F\left(\zeta \frac{\partial}{\partial x^i}\right)(p) = X^i(p) F\left(\zeta \frac{\partial}{\partial x^i}\right)(p) \\ = Y^i(p) F\left(\zeta \frac{\partial}{\partial x^i}\right)(p) = F(Y)(p). \end{aligned}$$

Hence $F(X)(p) = F(Y)(p)$.

Corollary. Let $B : \mathcal{V}(\mathcal{M}) \times \mathcal{V}(\mathcal{M}) \to C^\infty(\mathcal{M})$ be a bilinear form on $\mathcal{M}$, $p \in \mathcal{M}$, $X, \tilde{X}, Y, \tilde{Y} \in \mathcal{V}(\mathcal{M})$. If $X(p) = \tilde{X}(p), Y(p) = \tilde{Y}(p)$, then $B(X, Y)(p) = B(\tilde{X}, \tilde{Y})(p)$.

Pointwise definition of a bilinear form. Let B be a bilinear form on $\mathcal{M}, p \in \mathcal{M}, u, v \in T_p\mathcal{M}$. Define

$$B(u, v) := B(X, Y)(p),$$

where $X, Y \in \mathcal{V}(\mathcal{M}), X(p) = u, Y(p) = v$. From the previous corollary, we know $B(u, v)$ is well-defined. Then $B : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ is a biliear operator on $T_p\mathcal{M}$.

Local definition of a bilinear form. Let Ω open in $\mathcal{M}, X, Y \in \mathcal{V}(\Omega)$. Define $B : \mathcal{V}(\Omega) \times \mathcal{V}(\Omega) \to C^\infty(\Omega)$ by

$$B(X, Y)(p) = B(X_p, Y_p), \quad p \in \Omega.$$

Then B is a bilinear form on Ω.

Proof. First we verify that $B(X, Y) \in C^\infty(\Omega)$. It is sufficient to verify $B(X, Y)$ smooth near p. Choose a cut-off function $\zeta \in C^\infty(\mathcal{M})$ such that $\zeta \equiv 1$ near p and $\mathrm{supp}\zeta \subset \Omega$. Then

$$B(X, Y) \equiv B(\zeta X, \zeta Y) \text{ near } p.$$

Hence $B(X, Y)$ is smooth near p. Moreover, since B is a bilinear operator on $T_p\mathcal{M}$ for each p, it is easy to see that B is bilinear on $\mathcal{V}(\Omega)$.

Components of a bilinear form. Let Ω be a coordinate neighborhood with coordinate $\varphi = \varphi(x)$. Write

$$B_{ij} := B\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad i, j = 1, \cdots, n.$$

B_{ij} is a component of B, since

$$X = X^i \frac{\partial}{\partial x^i}, Y = Y^j \frac{\partial}{\partial x^j} \implies B(X, Y) = X^i Y^j B_{ij}.$$

Let g be a bilinear form on $\mathcal{M}$. g is called symmetric if

$$g(u, v) = g(v, u), \quad \forall p \in \mathcal{M}, u, v \in T_p\mathcal{M}.$$

g is called positive if

$$\begin{gathered} g(u,u) \geq 0, \forall p \in \mathcal{M}, \forall u \in T_p(\mathcal{M}), \\ \text{and} \quad g(u,u) = 0 \iff u = 0. \end{gathered}$$

Definition of Riemannian manifolds

Definition: Riemannian manifold. Let g be a symmetric positive bilinear form on $\mathcal{M}$. Then g is called a Riemannian metric on $\mathcal{M}$ and $(\mathcal{M}, g)$ is a called a Riemannian manifold.

Proposition. Every smooth manifold has a Riemannian metric.

Proof. Let $\{\zeta_k\}_{k=1}^{\infty}$ be a partition of unity with open cover $\{\Omega_k\}_{k=1}^{\infty}$, where Ω_k is a coordinate neighborhood with coordinate φ_k. Define a bilinear form g_k on Ω_k by

$$g_k(X,Y) = (X^i - Y^i)^2, \quad X, Y \in \mathcal{V}(\Omega_k).$$

Define a bilinear form g on $\mathcal{M}$ by

$$g(X,Y) = \sum_{k=1}^{\infty} g_k(\zeta_k X, \zeta_k Y), \quad X, Y \in \mathcal{V}(\mathcal{M}).$$

Then g is obviously symmetric and nonnegative. If $g(X,Y) = 0$, then $g_k(\zeta_k X, \zeta_k Y) = 0$ for each k. Hence $\zeta_k X = \zeta_k Y$ for each $k \implies X = Y$. Hence g is Riemannian metric on $\mathcal{M}$.

Let $(\mathcal{M}, g)$ be a Riemannian manifold. Then g is a linear product on $T_p(\mathcal{M})$ for each $p \in \mathcal{M}$. Write

$$|u| := g(u,u)^{\frac{1}{2}}, \quad u \in T_p(\mathcal{M}),$$

and

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad i, j = 1, \cdots n.$$

Then $(g_{ij})_{n \times n}$ is a symmetric positive-definite matrix-valued smooth function.

Definition: smooth and pointwise smooth curve. Let $\gamma : [a,b] \to \mathcal{M}$. γ is a smooth curve on $\mathcal{M}$ if $\gamma = \tilde{\gamma}\big|_{[a,b]}$ for some $\tilde{\gamma} \in C^{\infty}((a-\varepsilon, b+\varepsilon); \mathcal{M}), \varepsilon > 0$.
Let $\gamma : [a,b] \to \mathcal{M}$ be continuous. γ is called a pointwise smooth curve on $\mathcal{M}$ is there exists a partition

$$a = a_0 < a_1 < \cdots < a_N = b$$

such that $\gamma\big|_{[a_i, a_{i+1}]}$ is smooth, $i = 1, \cdots, N$.

Definition: length of a curve. Let $\gamma : [a,b] \to \mathcal{M}$ be pointwise smooth. Define the length of γ by

$$L(\gamma) = L(\gamma; [a,b]) = \int_a^b |\gamma'(t)| \mathrm{d}t.$$

Suppose $\mathcal{M}$ be connected.

Definition: metric on a Riemannian manifold. Let $p, q \in \mathcal{M}$. Define

$$d(p,q) = \inf\{L(\gamma) : \gamma \text{ pointwise smooth from } p \text{ to } q\}.$$

Then d is a metric on $\mathcal{M}$.

Proof. We need to show that $d(p,q)>0$ for $p\neq q$. Let $\gamma:[a,b]\to\mathcal{M}$ with $\gamma(a)=p,\gamma(b)=q$. Let $\varphi: B_2\to\mathcal{M}$ be a coordinate such that $\varphi^{-1}(p)=0, q\notin \mathrm{Rg}\varphi$. Choose $c\in(a,b)$ such that

$$\gamma(c)\in\varphi(\partial B_1),\quad \gamma(t)\in\varphi(B_1),\forall t\in[a,c).$$

Write $\alpha=\gamma\big|_{[a,c]}$ and $\alpha^i=(\varphi^{-1}\circ\alpha)^i, i=1,\cdots,n$. Then

$$\begin{aligned}|\alpha'(t)|^2=&\left|(\alpha^i)'(t)\frac{\partial}{\partial x^i}\Big|_{\alpha(t)}\right|\\ =&(\alpha^i)'(t)(\alpha^j)'(t)g_{ij}(\alpha(t))\\ \geq&\lambda_1(\alpha(t))|\big((\alpha^1)',\cdots,(\alpha^n)'\big)(t)|\geq c_0^2,\end{aligned}$$

where $\lambda_1:\mathrm{Rg}\varphi\to\mathbb{R}$ is the smallest eigenvalue of $(g_{ij})_{n\times n}$, which has a posive infimum c_0^2 on $\varphi(\overline{B}_1)$. Then

$$L(\gamma)\geq L(\alpha)=\int_a^c|\alpha'(t)|\mathrm{d}t\geq(c-a)c_0.$$

Hence $d(p,q)\geq(c-a)c_0>0$.

Integration on a Riemannian Manifold

Integration on a coordinate neighborhood. Let Ω be a coordinate neighborhood Ω with coordinate $\varphi=\varphi(x)$. Set

$$C_0(\Omega):=\{f\in C(\Omega):\mathrm{supp}f\text{ compact in }\Omega\}$$

and

$$V(Q)=\left(\det(g_{ij}(p))_{n\times n}\right)^{\frac{1}{2}}$$

where $Q=\{t^i\frac{\partial}{\partial x^i}\big|_p:0\leq t^i\leq1,i=1,\cdots,n\}$. Define

$$I_\Omega(f):=\int_{\varphi^{-1}(\Omega)}(f\circ\varphi)\left(\det(g_{ij})_{n\times n}\right)^{\frac{1}{2}}\circ\varphi.$$

The integral $I_\Omega(f)$ is independent of the coordinate $\varphi=\varphi(x)$.

Proof. Suppose there is another coordinate $\psi=\psi(y)$ on Ω. Write

$$\tilde{g}_{kl}=g\left(\frac{\partial}{\partial y^k},\frac{\partial}{\partial y^l}\right),\quad k,l=1,\cdots,n.$$

Since

$$\frac{\partial}{\partial x^i}=\frac{\partial(\psi^{-1})^k}{\partial x^i}\frac{\partial}{\partial y^k},$$

we have

$$g_{ij}=\frac{\partial(\psi^{-1})^k}{\partial x^i}\frac{\partial(\psi^{-1})^l}{\partial x^j}\tilde{g}_{kl},\quad i,j=1,\cdots,n.$$

Hence

$$(g_{ij}) = \nabla(\psi^{-1})^T(\tilde{g}_{kl})\nabla(\psi^{-1}),$$

where $\nabla(\psi^{-1}) = (\frac{\partial(\psi^{-1})^i}{\partial x^j}) = \nabla(\psi^{-1}\circ\varphi)\circ\varphi^{-1}$. Thus

$$(\det(g_{ij}))^{\frac{1}{2}} = |\det\nabla(\psi^{-1})|(\det(g_{kl}))^{\frac{1}{2}}.$$

Then by computation, we get

$$\begin{aligned}&\int_{\psi^{-1}(\Omega)}(f\circ\psi)(\det(\tilde{g}_{kl}))^{\frac{1}{2}}\circ\psi\\ =&\int_{\varphi^{-1}(\Omega)}(f\circ\varphi)(\det(\tilde{g}_{kl}))^{\frac{1}{2}}\circ\varphi|\det\nabla(\psi^{-1}\circ\varphi)|\\ =&\int_{\varphi^{-1}(\Omega)}(f\circ\varphi)\left(\det(g_{ij})_{n\times n}\right)^{\frac{1}{2}}\circ\varphi.\end{aligned}$$

Integral of continuous functions with compact support. Suppose $f\in C_0(\mathcal{M})$. Then there exist $\xi_1,\cdots,\xi_N\in C_0^\infty(\mathcal{M};[0,1])$ and coordinate neighborhood $\Omega_1,\cdots,\Omega_N$ such that $\mathrm{supp}\xi_i\subset\Omega_i,\forall i$ and $\sum\limits_{i=1}^N\xi_i=1$ on $\mathrm{supp}f$. Define

$$I(f)=\sum_{i=1}^N I_{\Omega_i}(\xi_i f).$$

It is easy to see that $I(f)$ is independent of the choose of ξ_i and Ω_i.

Integral of general functions on $\mathcal{M}$. Since $I:C_0(\mathcal{M})\to\mathbb{R}$ is a non-negative linear functional, by Riesz Representation Theorem, there exists a unique regular Borel measure V (called the volume measure) such that

$$I(f)=\int_{\mathcal{M}} f\mathrm{d}V,\quad \forall f\in C_0(\mathcal{M}).$$

Connections

Affine Connections

Definition: Affine connections. Let $\mathcal{M}$ be a smooth manifold. $D:\mathcal{V}(\mathcal{M})\times\mathcal{V}(\mathcal{M})\to\mathcal{V}(\mathcal{M})$ such that for $\forall f,g\in C^\infty(\mathcal{M}), X,Y,Z\in\mathcal{V}(\mathcal{M})$, we have
(i) $D_X(Y+Z)=D_XY+D_XZ$,
(ii) $D_X(fY)=(Xf)Y+fD_XY$,
(iii) $D_{fX+gY}Z=fD_XZ+gD_YZ$.
Then we call D an affine connection on $\mathcal{M}$.

Proposition. Every manifold has affine connections.

Proof. Choose a partition of unity $\{\zeta_k\}_{k=1}^\infty$, each $\mathrm{supp}\zeta_k$ contained in a coordinate neighborhood Ω_k. Let $\varphi_k=\varphi_k(x)$ be a coordinate on Ω_k. Define $D^k:\mathcal{V}(\Omega_k)\times\mathcal{V}(\Omega_k)\to\mathcal{V}(\Omega_k)$ by

$$D_X^k Y(p) = \langle X(p), Y^i\rangle \frac{\partial}{\partial x^i}(p), \quad p\in\Omega_k, X, Y \in \mathcal{V}(\Omega_k).$$

Then D_k is an affine connection on Ω_k. Define $D : \mathcal{V}(\mathcal{M}) \times \mathcal{V}(\mathcal{M}) \to \mathcal{V}(\mathcal{M})$ by

$$D_X Y = \sum_{k=1}^{\infty} D_X^k(\zeta_k Y), \quad X, Y \in \mathcal{V}(\mathcal{M}).$$

Then D is an affine connection on $\mathcal{M}$.

Proposition. Let $X, \tilde{X}, Y, \tilde{Y} \in \mathcal{V}(\mathcal{M}), p \in \mathcal{M}$.
(i) $X(p) = \tilde{X}(p) \implies D_X Y(p) = D_{\tilde{X}} Y(p)$.
(ii) $Y \equiv \tilde{Y}$ near $p \implies D_X Y(p) = D_{\tilde{X}} Y(p)$.

According to the proposition, we can define D on $\mathcal{V}(\Omega)$ where Ω is an open subset of $\mathcal{M}$ by

$$D_v X := D_V \tilde{X}(p), \quad V \in \tilde{X} \in \mathcal{V}(\mathcal{M}), V(p) = v, v \in T_p\Omega.$$

Then $D : T_p\Omega \times \mathcal{V}(\Omega) \to T_p\Omega$. And we have

$$\begin{gathered} D_v(\alpha X + \beta Y) = \alpha D_v X + \beta D_v Y, \\ D_v(fX) = \langle v, f\rangle X(p) + f(p) D_v X, \\ D_{\alpha v + \beta w} X = \alpha D_v X + \beta D_w X, \end{gathered}$$

for $\forall \alpha, \beta \in \mathbb{R}, v, w \in T_p\Omega, X, Y \in \mathcal{V}(\Omega), f \in C^\infty(\Omega)$.

We can also define $D : \mathcal{V}(\Omega) \times \mathcal{V}(\Omega) \to \mathcal{V}(\Omega)$ by

$$D_X Y(p) := D_{X(p)} Y, \quad p \in \Omega.$$

Then D is an affine connection on Ω.

For a local coordinate $\varphi = \varphi(x)$, write

$$D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}, \quad i, j = 1, \cdots, n.$$

We call Γ_{ij}^k the connection coefficients. For $X = X^i \frac{\partial}{\partial x^i}, Y = Y^j \frac{\partial}{\partial x^j}$, we have

$$D_X Y = X^i D_{\frac{\partial}{\partial x^i}} \left(Y^j \frac{\partial}{\partial x^j}\right) = \left(X^i \frac{\partial Y^k}{\partial x^i} + X^i Y^j \Gamma_{ij}^k\right) \frac{\partial}{\partial x^k}.$$

Thus the affine connection D is locally decided by its connection coefficients Γ_{ij}^k.

Proposition. Let D be an affine connection on smooth manifold $\mathcal{M}$, $\gamma : (a, b) \to \mathcal{M}$ be a smooth surve such that $\gamma(t_0) = p \in \mathcal{M}, \gamma'(t_0) = v \in T_p(\mathcal{M})$. Then for $X \in \mathcal{V}(\mathcal{M})$, $D_v X$ only depends on $X \circ \gamma$.

Proof.

$$\begin{aligned} D_v X &= D_v(X^i \frac{\partial}{\partial x^i}) = v(X^i) \frac{\partial}{\partial x^i}(p) + X^i(p) D_v \frac{\partial}{\partial x^i} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t_0} (X^i \circ \gamma) \frac{\partial}{\partial x^i}(p) + X^i \circ \gamma(t_0) D_v \frac{\partial}{\partial x^i}. \end{aligned}$$

Definition: vector field along a curve. Let $\gamma : (a, b) \to \mathcal{M}$ and $X : (a, b) \to T\mathcal{M}$ be smooth such that

$$X(t) \in T_{\gamma(t)}\mathcal{M}, \quad t \in (a, b).$$

Then we call X a smooth vector field along the curve γ. Let $\mathcal{V}(\gamma)$ be the set of all smooth vector fields along γ.

Examples. (i) $\gamma' \in \mathcal{V}(\gamma)$.
(ii) $X \in \mathcal{V}(\mathcal{M}) \implies X \circ \gamma \in \mathcal{V}(\gamma)$.
(iii) $f \in C^\infty(a,b), p \in \mathcal{M}, v \in T_p\mathcal{M}$, define $X(t) = f(t)v, t \in (a,b)$. Then $X \in \mathcal{V}(\gamma)$ with $\gamma(t) = p, t \in (a,b)$.

Proposition. Let D be an affine connection on a smooth manifold $\mathcal{M}$ and $\gamma : (a,b) \to \mathcal{M}$ be smooth. Then there exists a unique operator $\frac{D}{dt} : \mathcal{V}(\gamma) \to \mathcal{V}(\gamma)$ satisfying
(i) $\frac{D}{dt}(X+Y) = \frac{DX}{dt} + \frac{DY}{dt}, \quad \forall X, Y \in \mathcal{V}(\gamma)$.
(ii) $\frac{D}{dt}(fX) = f'X + f\frac{DX}{dt}, \quad \forall f, \in C^\infty((a,b)), X \in \mathcal{V}(\gamma)$.
(iii) If $X \in \mathcal{V}(\gamma)$ and there exists a smooth vector field $\tilde{X}$ in a neighborhood of $p = \gamma(t_0)$ such that $\tilde{X} \circ \gamma = X$ near t_0, then $\frac{DX}{dt}(t_0) = D_{\gamma'(t_0)}\tilde{X}$.

Remark. We often write $\frac{dX}{dt}$ instead of $\frac{DX}{dt}$.

Proof. We will prove the proposition by computation.

$$\begin{aligned}
\frac{dX}{dt} &= \frac{d}{dt}\left(X^i \frac{\partial}{\partial x^j}\right) \\
&= \frac{dX^j}{dt}\frac{\partial}{\partial x^j} + X^j \frac{d}{dt}\frac{\partial}{\partial x^j} \\
&= \frac{dX^k}{dt}\frac{\partial}{\partial x^k} + X^j D_{\gamma'}\frac{\partial}{\partial x^j} \\
&= \frac{dX^k}{dt}\frac{\partial}{\partial x^k} + X^j (\gamma^i)'\Gamma^k_{ij}\frac{\partial}{\partial x^k} \\
&= \left(\frac{dX^k}{dt} + \Gamma^k_{ij}\frac{d\gamma^i}{dt}X^j\right)\frac{\partial}{\partial x^k},
\end{aligned}$$

where $X = X^i \frac{\partial}{\partial x^j}, \gamma' = (\gamma^i)' \frac{\partial}{\partial x^i}$.

Examples. (i) $\frac{d^2\gamma}{dt^2} = \left(\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\right)\frac{\partial}{\partial x^k}$.
(ii) $v \in T_p\mathcal{M}, \gamma(t) = p, X(t) = f(t)v \implies \frac{dX}{dt} = f'v$.

Riemannian Connection

For Riemannian manifold $(\mathcal{M}, g)$, we write $\langle\cdot,\cdot\rangle = g(\cdot,\cdot)$.

Definition: metric connection. Let D be an affine connection on a Riemannian manifold $(\mathcal{M}, g)$. D is called a metric connection if

$$X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle, \quad \forall X, Y, Z \in \mathcal{V}(\mathcal{M}).$$

Proposition. Let D be a metric connection.
(i) $v(\langle X, Y\rangle) = \langle D_v X, Y(p)\rangle + \langle X(p), D_v(Y)\rangle$ for $v \in T_p\mathcal{M}$, X, Y smooth vector fields near p.
(ii) $\frac{d}{dt}\langle X, Y\rangle = \langle \frac{dX}{dt}, Y\rangle + \langle X, \frac{dY}{dt}\rangle$, for $\gamma : (a,b) \to \mathcal{M}$ smooth and $X, Y \in \mathcal{V}(\gamma)$.

Proof. (i) Choose smooth vector field V near p such that $V(p) = v$. Then

$$
\begin{aligned}
v(\langle X, Y\rangle) &= V\langle X, Y\rangle(p) \\
&= \langle D_V X, Y\rangle(p) + \langle X, D_V Y\rangle(p) \\
&= \langle D_v X, Y(p)\rangle + \langle X(p), D_v(Y)\rangle.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
&\frac{d}{dt}\langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle \\
=&\frac{d}{dt}\left(\langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle \circ \gamma\right) \\
=&\gamma'(t)\left(\langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle\right) \\
=&\langle D_{\gamma'}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle + \langle\frac{\partial}{\partial x^i}, D_{\gamma'}\frac{\partial}{\partial x^j}\rangle \\
=&\langle\frac{d}{dt}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle + \langle\frac{\partial}{\partial x^i}, \frac{d}{dt}\frac{\partial}{\partial x^j}\rangle \\
\Longrightarrow &\frac{d}{dt}\langle X, Y\rangle \\
=&\frac{dX^i}{dt}Y^j\langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle + X^i\frac{dY^j}{dt}\langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle + X^iY^j\frac{d}{dt}\langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle \\
=&\langle\frac{dX^i}{dt}\frac{\partial}{\partial x^i}, Y\rangle + \langle X, \frac{dY^j}{dt}\frac{\partial}{\partial x^j}\rangle + \langle X^i\frac{d}{dt}\frac{\partial}{\partial x^i}, Y\rangle + \langle X, Y^j\frac{d}{dt}\frac{\partial}{\partial x^j}\rangle \\
=&\langle\frac{d}{dt}X, Y\rangle + \langle X, \frac{d}{dt}Y\rangle.
\end{aligned}
$$

Definition: symmetric connection. An affine connection D on a smooth manifold $\mathcal{M}$ is called symmetric if

$$
D_X Y - D_Y X = [X, Y], \quad \forall X, Y \in \mathcal{M},
$$

which is locally equivalent to $\Gamma^k_{ij} = \Gamma^k_{ji}, \forall i, j, k = 1, \cdots, n.$

Definition: Riemannian connection. A symmetric metric connection D on a Riemannian manifold $\mathcal{M}$ is called Riemannian connection.

Example. The Riemannian connection on S^n is

$$
\nabla_X Y(p) = T_p(D_{\tilde{X}}\tilde{Y}(p)), \quad X, Y \in \mathcal{V}(S^n),
$$

where $\tilde{X}, \tilde{Y} \in \mathcal{V}(\mathbb{R}^{n+1}), \tilde{X}\big|_{S^n} = X, \tilde{Y}\big|_{S^n} = Y, T_p(x) = x - \langle x, p\rangle p, x \in \mathbb{R}^{n+1}, p \in S^n$ be the orthogonal projection from $\mathbb{R}^{n+1}$ to T_pS^n.

Proof. For $x \in \mathbb{R}^{n+1}, y \in T_pS^n$, we have $\langle T_px, T_py\rangle = \langle T_px, y\rangle = \langle x, y\rangle$. Then

$$
\begin{aligned}
X\langle Y, Z\rangle &= \tilde{X}\langle\tilde{Y}, \tilde{Z}\rangle = \langle D_{\tilde{X}}\tilde{Y}, \tilde{Z}\rangle + \langle\tilde{Y}, D_{\tilde{X}}\tilde{Z}\rangle \\
&= \langle T(D_{\tilde{X}}\tilde{Y}), Z\rangle + \langle Y, T(D_{\tilde{X}}\tilde{Z})\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.
\end{aligned}
$$

For $\tilde{X} = \tilde{X}^i\frac{\partial}{\partial x^i}, \tilde{Y} = \tilde{Y}^i\frac{\partial}{\partial x^i}$, we have

$$
\langle\tilde{X}(p), p\rangle = \langle\tilde{Y}(p), p\rangle = 0, \quad \forall p \in S^n,
$$

since $\tilde{X}\big|_{S^n} = X \in \mathcal{V}(S^n), \tilde{Y}\big|_{S^n} = Y \in \mathcal{V}(S^n)$. Then we claim that

$$
T([\tilde{X}, \tilde{Y}]) = [X, Y].
$$

For simplicity, we only prove the formula at $p = e_{n+1}$, when $T_pS^n = \text{span}\{\frac{\partial}{\partial x^i}\}_{i=1}^n$. We have $T_p(x) = x - x^{n+1}e_{n+1} = (x^1, \cdots, x^n, 0)$. Then $\tilde{X}^{n+1}(p) = \tilde{Y}^{n+1}(p) = 0$. Thus

$$
\begin{aligned}
T_p([\tilde{X}, \tilde{Y}](p)) &= T_p\left(\sum_{i,j=1}^{n+1}\left(\tilde{X}^i\frac{\partial \tilde{Y}^j}{\partial x^i} - \tilde{Y}^i\frac{\partial \tilde{X}^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}\right) \\
&= \sum_{i,j=1}^{n}\left(X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j} \\
&= [X, Y](p).
\end{aligned}
$$

Finally, we get

$$\nabla_X Y - \nabla_Y X = T(D_{\tilde{X}}\tilde{Y} - D_{\tilde{Y}}\tilde{X}) = T([\tilde{X}, \tilde{Y}]) = [X, Y].$$

Theorem. The Riemanian connection exists uniquely, the connection coefficients of which are given by

$$\Gamma_{ij}^k = \Gamma_{ij,m}g^{mk}, \quad i, j, k \in \{1, \cdots, n\},$$

where

$$
\begin{aligned}
\Gamma_{ij,m} &= \frac{1}{2}\left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right), \quad i, j, m \in \{1, \cdots, n\}, \\
(g^{ij})_{n\times n} &= (g_{ij})_{n\times n}^{-1}, \quad g_{ij} = \langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle, \quad i, j \in \{1, \cdots, n\}.
\end{aligned}
$$

Proof. The affine connection D given by $\Gamma_{ij}^k = \Gamma_{ij,m}g^{mk}$ is obviously symmetric since $\Gamma_{ij}^k = \Gamma_{ji}^k$. Moreover,

$$
\begin{aligned}
&X\langle Y, Z\rangle \\
=&X^i\partial_i\langle Y^j\partial_j, Z^k\partial_k\rangle \\
=&X^i(\partial_iY^jZ^k + Y^j\partial_iZ^k)g_{jk} + X^iY^jZ^k\partial_ig_{jk}, \\
&\langle D_XY, Z\rangle + \langle Y, D_XZ\rangle \\
=&(X^i\partial_iY^l + X^iY^j\Gamma_{ij}^l)Z^kg_{lk} + (X^i\partial_iZ^l + X^iZ^j\Gamma_{ij}^l)Y^kg_{lk}.
\end{aligned}
$$

Hence

$$\langle D_XY, Z\rangle + \langle Y, D_XZ\rangle - X\langle Y, Z\rangle = X^i(Y^jZ^k + Y^kZ^j)\Gamma_{ij}^lg_{lk} - X^iY^jZ^k\partial_ig_{jk}.$$

Since $\Gamma_{ij}^lg_{lk} = \Gamma_{ij,m}g^{ml}g_{lk} = \Gamma_{ij,m}\delta_{mk} = \Gamma_{ij,k}$, we have

$$
\begin{aligned}
&\langle D_XY, Z\rangle + \langle Y, D_XZ\rangle - X\langle Y, Z\rangle \\
=&\frac{1}{2}X^iY^jZ^k(\Gamma_{ij,k} + \Gamma_{ik,j}) - X^iY^jZ^k\partial_ig_{jk} \\
=&\frac{1}{2}X^iY^jZ^k(\partial_jg_{ik} + \partial_ig_{jk} - \partial_kg_{ij} + \partial_kg_{ij} + \partial_ig_{kj} - \partial_jg_{ik} - 2\partial_ig_{jk}) = 0.
\end{aligned}
$$

Thus the connection is exactly a Riemannian connection. Next, we will prove the uniqueness. We claim that if a Riemannian connection D has coefficients Γ_{ij}^k, then $\Gamma_{ij}^k = \Gamma_{ij,m}g^{mk}$. Indeed,

$$
\begin{aligned}
\partial_jg_{im} &= \partial_j\langle\partial_i, \partial_m\rangle = \langle D_{\partial_j}\partial_i, \partial_m\rangle + \langle\partial_i, D_{\partial_j}\partial_m\rangle, \\
\partial_ig_{jm} &= \langle D_{\partial_i}\partial_j, \partial_m\rangle + \langle\partial_j, D_{\partial_i}\partial_m\rangle, \\
\partial_mg_{ij} &= \langle D_{\partial_m}\partial_i, \partial_j\rangle + \langle\partial_i, D_{\partial_m}\partial_j\rangle.
\end{aligned}
$$

Thus

$$\Gamma_{ij,m} = \langle D_{\partial_i}\partial_j, \partial_m\rangle = \Gamma^l_{ij} g_{lm} \implies \Gamma_{ij,m} g^{mk} = \Gamma^k_{ij}.$$

Example. Let $H^2 = \{(x,y) : x \in \mathbb{R}, y > 0\}, g_{ij}(x,y) = y^{-2}\delta_{ij}$. Then

$$\partial_x g_{ij} = 0, \quad \partial_y g_{ij} = -2y^{-3}\delta_{ij}.$$

Hence

$$\Gamma^k_{ij} = -y^{-1}(\delta_{im} + \delta_{2j} + \delta_{jm}\delta_{2i} - \delta_{ij}\delta_{2m})\delta_m = y^{-1}(\delta_{ij}\delta_{2k} - \delta_{jk}\delta_{2i} - \delta_{ik}\delta_{2j}).$$

Thus

$$(\Gamma^1_{ij}) = y^{-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (\Gamma^2_{ij}) = y^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Geodesics

Some Preliminaries

Definition: vector fields along a surface. Let $\mathcal{M}$ be a manifold, $f = f(s,t) \in C^\infty((a,b)\times(c,d);\mathcal{M})$ is called a smooth surface on $\mathcal{M}$. $X = X(s,t) \in C^\infty((a,b)\times(c,d);T\mathcal{M})$ is called a smooth vector field along f is $X(s,t) \in T_{f(s,t)}\mathcal{M}, \forall s \in (a,b), t \in (c,d)$.

For a function $F = F(s,t) : A \times B \to C$, we write $F_s(t) = F(s,t), F_t(s) = F(s,t)$.

Set

$$\frac{\partial f}{\partial t} = \frac{df_s}{dt}, \quad \frac{\partial f}{\partial s} = \frac{df_t}{ds}, \quad \frac{\partial X}{\partial t} = \frac{dX_s}{dt}, \quad \frac{\partial X}{\partial s} = \frac{dX_t}{ds}$$

are vector fields along f.

Proposition. $\frac{\partial^2 f}{\partial s\partial t} = \frac{\partial^2 f}{\partial t\partial s}$.

Proof. Write $f^i = (\varphi^{-1}\circ f)^i, X = X^i\partial_i$. Then

$$\frac{\partial f}{\partial s} = \frac{\partial f^i}{\partial s}\partial_i, \quad \frac{\partial f}{\partial t} = \frac{\partial f^i}{\partial t}\partial_i,$$
$$\frac{\partial X}{\partial s} = \left(\frac{\partial X^k}{\partial s} + \Gamma^k_{ij}\frac{\partial f^i}{\partial s}X^j\right)\partial_k.$$

Hence

$$\frac{\partial}{\partial s}\frac{\partial f}{\partial t} = \left(\frac{\partial^2 f^k}{\partial s\partial t} + \Gamma^k_{ij}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial t}\right)\partial_k = \frac{\partial}{\partial t}\frac{\partial f}{\partial s}.$$

Remark. From the proof above, we see that in general $\frac{\partial^2 X}{\partial s\partial t} \neq \frac{\partial^2 X}{\partial t\partial s}$.

Definition: local diffeomorphism. $Let \mathcal{M}, \mathcal{N}$ be two manifolds, $F \in C^\infty(\mathcal{M};\mathcal{N})$. F is a local diffeomorphism if $\forall p \in \mathcal{M}$, there exist a neighborhood U of p and V of $f(p)$ such that $F : U \to V$ is a diffeomorphism\dots

Remark. $F \in C^\infty(\mathcal{M};\mathcal{N})$ is a local diffeomorphism if and only if $\forall p \in \mathcal{M}$, dF_p is bijective.

Definition: Riemannian isometry. Let $\mathcal{M},\mathcal{N}$ be two Riemannian manifolds, $F \in C^\infty(\mathcal{M};\mathcal{N})$ be a diffeomorphism. F is called a Riemannian isometry if

$$\langle dF_p(u), dF_p(v)\rangle_{\mathcal{N}} = \langle u, v\rangle_{\mathcal{M}}, \quad \forall u, v \in T_p\mathcal{M}, p \in \mathcal{M}. \tag{$*$}$$

Definition: local Riemmanian isometry. Let $F \in C^\infty(\mathcal{M};\mathcal{N})$. F is called a local Riemannian isometry if $\forall p \in \mathcal{M}$, there exist a neighborhood U of p and V of $F(p)$ such that $F : U \to V$ is a Riemannian isometry.

Remark. $F \in C^\infty(\mathcal{M};\mathcal{N})$ is a local Riemannian isometry if F satisfies $(*)$.

Example. $F : \mathbb{R} \to \mathbb{S}^1, F(t) = e^{it}$ is a local Riemannian isometry.

Proposition. Let $F : \mathcal{M} \to \mathcal{N}$ is a local Riemannian isometry. Then
(i) $L(F(\gamma)) = L(\gamma)$ for any curve γ on $\mathcal{M}$.
(ii) $d_{\mathcal{N}}(F(p), F(q)) \le d_{\mathcal{M}}(p, q), \quad \forall p, q \in \mathcal{M}$.

Geodesic and Expotential Map

Definition: geodesic. Let $\gamma : (a, b) \to \mathcal{M}$ be a smooth curve. The geodesic equation is

$$\frac{d^2\gamma}{dt} = 0,$$

which is locally equivalent to

$$\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0, \quad \forall k = 1, \cdots, n.$$

We say a non-constant curve γ is a geodesic if γ satisfies the geodesic equation. If $\gamma \in C^\infty([a, b];\mathcal{M})$ can be extended to be a geodesic, we call γ a geodesic segment.

Proposition. Let γ be a geodesic.
(i) $\alpha(t) = \gamma(at + b)$ with $a, b \in \mathbb{R}, a \ne 0$ is also a geodesic.
(ii) $|\gamma'| \equiv \text{constant}$.

Proof. (i) Obviously. (ii) $\frac{d}{dt}|\gamma'|^2 = \frac{d}{dt}\langle\gamma', \gamma'\rangle = 2\langle\gamma'', \gamma'\rangle = 0$.

For $v \in T\mathcal{M}$, let $\Gamma(v, \cdot) = \Gamma(v, t)$ be the unique solution of

$$\begin{cases} \gamma'' = 0, \\ \gamma'(0) = v. \end{cases}$$

Let (a_v, b_v) with $-\infty \le a_v < b_v \le \infty$ be the maximum domain of $\Gamma(v, \cdot)$. Then

$$\Gamma \in C^\infty(\Omega;\mathcal{M}), \quad \Gamma(kv, t) = \Gamma(v, kt), \quad \forall v \in T\mathcal{M}, k, t \in \mathbb{R},$$

where

$$\Omega = \{(v, t) : v \in T\mathcal{M}, t \in (a_v, b_v)\}$$

is an open subset of $T\mathcal{M} \times \mathbb{R}$.

Example. On the sphere S^n, let $p \in S^n, v \in T_pS^n$. Then

$$\Gamma(v,t) = (\cos|v|t)p + (\sin|v|t)\frac{v}{|v|}.$$

Definition: expotential map. Let $D = \{v \in T\mathcal{M} : 1 \in (a_v, b_v)\}$ be open in $T\mathcal{M}$. Define

$$\exp : D \to \mathcal{M}, \exp(v) = \Gamma(v,1).$$

Then $\exp \in C^\infty(D;\mathcal{M})$ is called expotential map. Moreover, $\Gamma(v,t) = \exp(tv)$.

Let $D_p = D \cap T_p\mathcal{M}$ be a star-shaped neighborhood of 0_p in $T_p\mathcal{M}$. Write

$$\exp_p(v) := \exp(v), \quad v \in T_p\mathcal{M}.$$

Then $d(\exp_p)_v : T_v D_p (= T_p\mathcal{M}) \to T_{\exp_p(v)}\mathcal{M}$. For $v = 0_p$,

$$d(\exp_p)_{0_p} = I_{T_p\mathcal{M}},$$

since $d(\exp_p)_{0_p}(v) = \frac{d}{dt}\big|_0 \exp_p(tv) = v$.

Definition: geodesic neighborhood. By the inverse mapping theorem, there exists a neighborhood U of 0_p such that

$$\exp_p : U \to \exp_p(U)$$

is a diffeomorphism. We call $\exp_p(U)$ a geodesic neighborhood of p. If in addition $U = B_R(0_p) \subset T_p\mathcal{M}$, write

$$B_R(p) := \exp_p(B_R(0_p))$$

be the geodesic ball. For $r \in (0,R)$, we also write

$$S_r(p) := \exp_p(\partial B_r(0_p)),$$

and radial geodesic $\exp_p(tv), 0 \le t \le 1$ with $|v| = r$.
There also exists a neighborhood of p and $\delta > 0$ such that $\forall q \in U$,

$$\exp_q : B_r(0_q) \to \exp_q(B_r(0_q))$$

is a diffeomorphism satisfying $U \subset B_r(0_q)$. U is called total geodesic neighborhood.

Minimizing Properties

Gauss's Lemma. Let $p \in \mathcal{M}, v \in \mathrm{Dom}(\exp_p)$. Then

$$\langle d(\exp_p)_v(v), d(\exp_p)_v(w)\rangle = \langle v, w\rangle, \quad \forall w \in T_p\mathcal{M}.$$

Proof. Select a smmoth curve $\alpha : (-\varepsilon,\varepsilon) \to \mathrm{Dom}(\exp_p)$ such that $\alpha(0) = v, \alpha'(0) = w$ and write $f(s,t) = \exp_p(t\alpha(s))$. Then

$$\partial_s f = d(\exp_p)_{t\alpha(s)}(t\alpha'(s)), \quad \partial_t f(s) = d(\exp_p)_{t\alpha(s)}(\alpha(s)).$$

Moreover,

$$
\begin{aligned}
\partial_t\langle\partial_s f,\partial_t f\rangle =&\langle\partial_{ts} f,\partial_t f\rangle+\langle\partial_s f,\partial_t^2 f\rangle\\
=&\langle\partial_{st} f,\partial_t f\rangle=\frac{1}{2}\partial_s\langle\partial_t f,\partial_t f\rangle\\
=&\frac{1}{2}\partial_s\langle\partial_t f,\partial_t f\rangle\big|_{t=0}=\frac{1}{2}\partial_s\langle\alpha(s),\alpha(s)\rangle\\
=&\langle\alpha'(s),\alpha(s)\rangle.
\end{aligned}
$$

Hence

$$
\langle\partial_s f,\partial_t f\rangle(s,t)=\langle\alpha'(s),\alpha(s)\rangle t.
$$

Taking $s=0, t=1$, we get

$$
\langle w,v\rangle=\langle\partial_s f(0,1),\partial_t f(0,1)\rangle=\langle d(\exp_p)_v(w),d(\exp_p)_v(v)\rangle.
$$

Theorem. Let $p\in\mathcal{M}, B_R(p)$ geodesic ball, $v\in B_R(0_p)\setminus\{0_p\}, q=\exp_p(v), \alpha:[0,1]\to\mathcal{M}$ a piecewise smooth curve such that $\alpha(0)=p,\alpha(1)=q$. Then
(i) $L(\alpha)\geq|v|$.
(ii) If $L(\alpha)=|v|$ and $|\alpha'|=\text{constant}$, then $\alpha(t)=\exp_p(tv), t\in[0,1]$. Hence $|v|=d(p,q)$.

Proof. (i) Assume $\operatorname{Ran}\alpha\subset B_R(p)$ and $\alpha(t)\neq p,\forall t\in(0,1]$. Let $\beta=\exp_p^{-1}\circ\alpha$. Then $\beta(0)=0_p,\beta(1)=v,\beta(t)\neq 0, t\in(0,1]$. Set

$$
\begin{aligned}
\beta'_\perp=\frac{\langle\beta',\beta\rangle}{\langle\beta,\beta\rangle}\beta,\quad \beta'_\parallel=\beta'-\beta'_\perp,\\
\alpha'_\perp=d(\exp_p)_\beta(\beta'_\perp),\quad \alpha'_\parallel=d(\exp_p)_\beta(\beta'_\parallel).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\alpha'=\alpha'_\perp+\alpha'_\parallel,\quad\langle\alpha'_\perp,\alpha'_\parallel\rangle=0\\
\Longrightarrow&|\alpha'|=\left(|\alpha'_\perp|^2+|\alpha'_\parallel|^2\right)^{\frac{1}{2}}\geq|\alpha'_\perp|=|\beta'_\perp|=|\beta|^{-1}|\langle\beta',\beta\rangle|=\big|\partial_t|\beta|\big|\\
\Longrightarrow&L(\alpha)=\int_0^1|\alpha'|dt\geq\int_0^1\big|\partial_t|\beta|\big|dt\geq\int_0^1\partial_t|\beta|dt=|v|.
\end{aligned}
$$

(ii) Suppose $L(\alpha)=|v|,|\alpha'|=\text{constant}$. Since $|\alpha'|=|\alpha'_\perp|$, we have $\alpha'_\parallel=0$. Hence $\beta'_\parallel=0$. Thus $|\beta'|=|\beta'_\perp|=|\alpha'|=|v|$. Hence $|\int_0^1\beta' dt|=\int_0^1|\beta'|dt=|v|$. Write

$$
e=\left|\int_0^1|\beta'|dt\right|^{-1}\int_0^1\beta' dt=|v|^{-1}\int_0^1\beta' dt.
$$

Then $|e|=1$. Observe that

$$
\int_0^1(|\beta'|-\langle\beta',e\rangle)dt=0,\quad|\beta'|-\langle\beta',e\rangle\geq 0.
$$

Hence $|\beta'|=\langle\beta',e\rangle$. By the Cauchy-Schwartz inequality, we get $\beta'\parallel e$, which implies $\beta'=v,\beta=vt,\alpha=\exp_p\beta=\exp_p(vt)$.

Cororllary1. Let $B_R(p)$ be a geodesic ball, $0<r<R, q\in S_r(p)$. Then $d(p,q)=r$.

Cororllary2. $B_R(p)=\{x\in\mathcal{M}:d(x,p)<R\}$.

Proof. For $q \in B_R(p)$, we have $q \in S_r(p)$ for some $r \in [0, R)$. Thus $d(p, q) = r < R$.
For $q \in \mathcal{M} \setminus B_R(p)$, given $\gamma : [0, 1] \to \mathcal{M}, \gamma(0) = p, \gamma(1) = q$ and $r \in (0, R)$, there exists $c \in [0, 1]$, $\gamma(c) \in S_r(p)$. Then $L(\gamma) \geq L(\gamma; [0, c]) \geq d(p, \gamma(c)) = r$. Let $r \to R$, we get $L(\gamma) \geq R$. Hence $d(p, q) \geq R$.

Cororllary3. Let $\gamma : [a, b] \to \mathcal{M}$ be a non-constant piecewise smooth curve such that $|\gamma'| = \text{constant}, L(\gamma) = d(\gamma(a), \gamma(b))$. Then γ is a geodesic segment.

Proof. Choose a geodesic ball $B_R(\gamma(a))$ and $r \in (0, R)$. Then there exists $c \in (a, b), \gamma(c) \in S_r(\gamma(a))$. Since $\gamma\big|_{[a,c]}$ satisfies the minimizing property, we get $\gamma\big|_{[a,c]}$ is a geodesic segment, which means $\gamma\big|_{[a,c]}$ can be extended to a geodesic $\gamma\big|_{(a-\varepsilon,a+\varepsilon)}$. Similarly we can extend γ at $\gamma(b)$ to $(b - \varepsilon, b + \varepsilon)$. Given $t \in (a, b)$, we also get γ is a geodesic in $(t, t + \varepsilon(t))$ for some $\varepsilon(t) > 0$. Then

$$[a, b] \subset (a - \varepsilon, a + \varepsilon) \cup (b - \varepsilon, b + \varepsilon) \cup \bigcup_{t\in(a,b)} (t, t + \varepsilon(t)).$$

Hence γ is a geodesic segment.

Hopf-Rinow Theorem

Theorem. Let $p \in \mathcal{M}$. Assume $\text{Dom}(\exp_p) = T_p\mathcal{M}$. Then for each $q \in \mathcal{M}, d(p, q) = r > 0$, there exists $v \in T_p\mathcal{M}, |v| = 1$ such that $q = \exp_p(rv)$.

Proof. Choose $B_{2\varepsilon}(p)$ such that $q \notin B_{2\varepsilon}(p)$. Then there exists $m \in S_\varepsilon(p)$ satisfying

$$d(m, q) = \inf_{x\in S_\varepsilon(p)} d(x, q).$$

Then $m = \exp_p(\varepsilon v)$ for some $v \in T_p\mathcal{M}, |v| = 1$. Set $\gamma(t) = \exp_p(tv), t \in \mathbb{R}$.
We claim that $\varepsilon + d(m, q) = d(p, q)$. Indeed, for $\alpha : [0, 1] \to \mathcal{M}, \alpha(0) = p, \alpha(1) = q$, we have $\alpha(s) \in S_\varepsilon(p)$ for some $s \in (0, 1)$. Then

$$L(\alpha) = L(\alpha; [0, s]) + L(\alpha; [s, 1]) \geq \varepsilon + d(m, q).$$

Hence $d(p, q) \geq \varepsilon + d(m, q)$. The "$\leq$" inequality is obvious.
Set

$$T = \{t \in [0, r] : t + d(\gamma(t), q) = d(p, q)\}.$$

We have $\varepsilon \in T \neq \varnothing$. Let $t_0 = \sup T$ and assume that $t_0 < r$. By definition, $t_0 \in T$. Write $p' = \gamma(t_0)$.
Repeating the procedure above, we get some $\varepsilon' > 0, m' \in S_{\varepsilon'}(p)$ such that

$$d(m', q) = \inf_{x\in S_{\varepsilon'}(p')} d(x, q),$$

and $v' \in T_{p'}\mathcal{M}, |v'| = 1, m' = \exp_{p'}(\varepsilon' v'), \varepsilon' + d(m', q) = d(p', q)$.
Write

$$\tilde{\gamma}(t) = \begin{cases} \exp_p(tv), & t \in [0, t_0], \\ \exp_{p'}((t - t_0)v'), & t \in (t_0, t_0 + \varepsilon']. \end{cases}$$

Then

$$\begin{aligned}&\varepsilon' + d(m', q) = d(p', q), t_0 + d(p', q) = d(p, q)\\ \implies & t_0 + \varepsilon' + d(m', q) = d(p, q)\\ \implies & L(\tilde{\gamma}) = d(p, q) - d(m', q) \le d(p, m')\\ \implies & L(\tilde{\gamma}) = d(p, m').\end{aligned}$$

By Corollary 3, $\tilde{\gamma}$ is a geodesic segment, since $|\tilde{\gamma}| \equiv 1$. We get $m' = \gamma(t_0 + \varepsilon')$ and $t_0 + \varepsilon' \in T$, which is a controdiction.
Thus $r = \sup T \in T$, and $r = d(p, q) = r + d(\gamma(r), q)$. Whence $\gamma(r) = q$.

Hopf-Rinow Theoren. Let $p \in \mathcal{M}$. Then the following are equivalent.
(i) $\mathrm{Dom}(\exp_p) = T_p\mathcal{M}$.
(ii) Each bounded closed subset of $\mathcal{M}$ is compact.
(iii) $\mathcal{M}$ is a complete metric space.
(iv) $\forall q \in \mathcal{M}, \mathrm{Dom}(\exp_q) = T_q\mathcal{M}$. (geodesically complete)
Moreover, each of (i)-(iv) implies
(v) $\forall q_1, q_2 \in \mathcal{M}, q_1 \neq q_2$, there exists some minimizing geodesic segement from q_1 to q_2.

Proof. (i) $\implies$ (ii). Since every bounded closed set is contained in some closed ball centered at p, it is sufficient to show that $\overline{B}_R(p) = \{x \in \mathcal{M} : d(x, p) \le R\}$ is compact for each $R > 0$. Choose $q \in B_R(p), d(p, q) = r < R$. By the previous theorem, we can find some $v \in T_p\mathcal{M}, |v| = 1, \exp_p(rv) = q$. Then $q \in \exp_p(\overline{B}_R(0_p))$. Hence $B_R(p) \subset \exp_p(\overline{B}_R(0_p))$. Taking the closure, we get $\overline{B}_R(p) \subset \exp_p(\overline{B}_R(0_p))$ which is homeomorphic to a closed ball in $\mathbb{R}^n$. Since a closed subset of a compact set is compact, we get $\overline{B}_R(p)$ is compact.
(ii) $\implies$ (iii). Obviously.
(iii) $\implies$ (iv). Let $\gamma : (a, b) \to \mathcal{M}$ solve the ODE

$$\begin{cases}\gamma'' = 0,\\ \gamma'(0) = v \in T_q\mathcal{M}, |v| = 1.\end{cases}$$

We need to verify that $a = -\infty, b = \infty$. Assume that $b < \infty$. Then $\forall \{t_i\} \subset (a, b), t_i \to b$, $\{\gamma(t_i)\}$ is a Cauchy sequence since

$$d(\gamma(t_i), \gamma(t_j)) \le |t_i - t_j|, \quad \forall i, j.$$

Then $\gamma(t_i)$ converges to some $m \in \mathcal{M}$. Thus

$$\gamma(t) \to m \text{ as } t \to b^-.$$

Define

$$\gamma(b) := \lim_{t \to b^-} \gamma(t) = m.$$

Then $\gamma : (a, b] \to \mathcal{M}$ is continuous. Choose a totally geodesic neighborhood (U, r) of $\gamma(b)$. Then for some $\varepsilon > 0$, $\gamma([b - \varepsilon, b]) \subset U$. Since U is a totally geodesic neighborhood, $U \subset B_r(\gamma(b - \varepsilon))$. There exists a unique gedesic α joint $\gamma(b - \varepsilon)$ and $\gamma(b)$. Then α extends γ, which implies $b = \infty$. Similiarly we can prove $a = -\infty$.
(iv) $\implies$ (i)(v). Obviously.

Curvature

Definition: curvature tensor. Let $\mathcal{M}$ be a Riemannian manifold. Define $R : \left(\mathcal{V}(\mathcal{M})\right)^3 \to \mathcal{V}(\mathcal{M})$ by

$$R(X,Y,Z) = D_Y D_X Z - D_X D_Y Z + D_{[X,Y]} Z.$$

Then R is a 4-tensor. We call R the curvature tensor of $\mathcal{M}$. R is C^∞-linear, since

$$\begin{aligned}
R(fX,Y,Z) &= D_Y D_{fX} Z - D_{fX} D_Y Z + D_{[fX,Y]} Z \\
&= D_Y (f D_X Z) - f D_X D_Y Z + D_{f[X,Y]-(Yf)X} Z \\
&= (Yf) D_X Z + f D_Y D_X Z - f D_X D_Y Z + f D_{[X,Y]} Z - (Yf) D_X Z \\
&= f D_Y D_X Z - f D_X D_Y Z + f D_{[X,Y]} Z = f R(X,Y,Z), \\
R(X,Y,fZ) &= D_Y D_X (fZ) - D_X D_Y (fZ) + D_{[X,Y]} (fZ) \\
&= D_Y ((Xf)Z + f D_X Z) - D_X ((Yf)Z + f D_Y Z) + ([X,Y]f)Z + f D_{[X,Y]} Z \\
&= (YXf)Z + (Xf) D_Y Z + (Yf) D_X Z + f D_Y D_X Z \\
&\quad - (XYf)Z - (Yf) D_X Z - (Xf) D_Y Z - f D_X D_Y Z + ([X,Y]f)Z + f D_{[X,Y]} Z \\
&= f D_Y D_X Z - f D_X D_Y Z + f D_{[X,Y]} Z = f R(X,Y,Z).
\end{aligned}$$

We also write

$$R(X,Y,Z,W) = \langle R(X,Y,Z), W \rangle \in C^\infty(\mathcal{M})$$

and

$$R(X,Y) = D_Y D_X - D_X D_Y + D_{[X,Y]} : \mathcal{V}(\mathcal{M}) \to \mathcal{V}(\mathcal{M}).$$

Moreover,

$$R(\partial_i, \partial_j) = D_{\partial_j} D_{\partial_i} - D_{\partial_i} D_{\partial_j}.$$

Theorem.
(i) $R(X,Y,Z,W) = -R(Y,X,Z,W)$.
(ii) $R(X,Y,Z,W) = -R(X,Y,W,Z)$.
(iii) $R(X,Y,Z) + R(Z,X,Y) + R(Y,Z,X) = 0$.
(iv) $R(X,Y,Z,W) = R(Z,W,X,Y)$.

Proof. (i) Obviously.
(ii) Observe that

$$\begin{aligned}
R(X,Y,Z,Z) =& \langle D_Y D_X Z, Z \rangle - \langle D_X D_Y Z, Z \rangle + \langle D_{[X,Y]} Z, Z \rangle \\
=& \frac{1}{2} YX \langle Z, Z \rangle - \langle D_X Z, D_Y Z \rangle \\
& - \frac{1}{2} XY \langle Z, Z \rangle + \langle D_Y Z, D_X Z \rangle \\
& + \frac{1}{2} [X,Y] \langle Z, Z \rangle = 0.
\end{aligned}$$

Thus

$$\begin{aligned}
0 =& R(X,Y,Z+W,Z+W) \\
=& R(X,Y,Z,Z) + R(X,Y,Z,W) + R(X,Y,W,Z) + R(X,Y,W,W) \\
=& R(X,Y,Z,W) + R(X,Y,W,Z).
\end{aligned}$$

(iii)

$$
\begin{aligned}
&D(X,Y,Z)+D(Z,X,Y)+D(Y,Z,X)\\
=&D_YD_XZ-D_XD_YZ+D_{[X,Y]}Z\\
&+D_XD_ZY-D_ZD_XY+D_{[Z,X]}Y\\
&+D_ZD_YX-D_YD_ZX+D_{[Y,Z]}X\\
=&D_Y[X,Z]+D_X[Z,Y]+D_Z[Y,X]-D_{[Y,X]}Z-D_{[X,Z]}Y-D_{[Z,Y]}X\\
=&[Y,[X,Z]]+[X,[Z,Y]]+[Z,[Y,X]]=0.
\end{aligned}
$$

(iv)

$$
\begin{aligned}
R(Z,W,X,Y)=&-R(W,X,Z,Y)-R(X,Z,W,Y)\\
=&R(W,X,Y,Z)+R(X,Z,Y,W)\\
=&-R(X,Y,W,Z)-R(Y,W,X,Z)-R(Z,Y,X,W)-R(Y,X,Z,W)\\
=&2R(X,Y,Z,W)+R(Y,W,Z,X)+R(Z,Y,W,X)\\
=&2R(X,Y,Z,W)-R(W,Z,Y,X)\\
=&2R(X,Y,Z,W)-R(Z,W,X,Y).
\end{aligned}
$$

Notation: For $F:\left(\mathcal{V}(\mathcal{M})\right)^4\to\mathcal{V}(\mathcal{M})$, write

$$\sigma F(X,Y,Z,W)=F(X,Y,Z,W)+F(Y,Z,X,W)+F(Z,X,Y,W).$$

Theorem.

$$D_XR(Y,Z,W)+D_ZR(X,Y,W)+D_YR(Z,X,W)=0,$$

where $D_XR(Y,Z,W)=D_X(R(Y,Z,W))-R(D_XY,Z,W)-R(Y,D_XZ,W)-R(Y,Z,D_XW)$.

Proof. Observe that

$$
\begin{aligned}
D_XR(Y,Z,W)=&D_X(R(Y,Z)W)-R(D_XY,Z)W-R(Y,D_XZ)W-R(Y,Z)D_XW\\
=&[D_X,R(Y,Z)]W-R(D_XY,Z)W-R(Y,D_XZ)W.
\end{aligned}
$$

Then

$$\sigma D_XR(Y,Z,W)=\sigma[D_X,R(Y,Z)]W-\sigma R(Y,D_XZ)W=-\sigma R(Y,Z)D_XW.$$

Since

$$R(Y,Z)=D_ZD_Y-D_YD_Z+D_{[Y,Z]}=[D_Z,D_Y]-D_{[Z,Y]},$$

we have

$$\sigma[D_X,R(Y,Z)]W=\sigma[D_X,[D_Z,D_Y]]W-\sigma[D_X,D_{[Z,Y]}]W=-\sigma[D_X,D_{[Z,Y]}]W.$$

Moreover,

$$
\begin{aligned}
\sigma R(D_XY,Z)W+\sigma R(Y,D_XZ)W=&\sigma R(D_ZX,Y)W-\sigma R(D_XZ,Y)W\\
=&\sigma R([Z,X],Y)W\\
=&\sigma[D_Y,D_{[Z,X]}]W-\sigma D_{[Y,[Z,X]]}W\\
=&\sigma[D_Y,D_{[Z,X]}]W.
\end{aligned}
$$

Hence

$$\sigma D_X R(Y,Z,W) = -\sigma[D_X, D_{[Z,Y]}]W - \sigma[D_Y, D_{[Z,X]}]W = \sigma[D_X, D_{[Y,Z]}]W - \sigma[D_Y, D_{[Z,X]}]W = 0.$$

Proposition.

$$R(\partial_i, \partial_j, \partial_k) = R^l_{ijk}\partial_l,$$

where

$$R^l_{ijk} = \partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}.$$

Thus $R(\partial_i, \partial_j, \partial_k, \partial_l) = R_{ijkl} = R^m_{ijk} g_{ml}$. By symmetric, we also have

$$R_{ijkl} = -R_{jikl}, \quad R_{ijkl} = -R_{ijlk}, \quad R_{ijkl} + R_{jkil} + R_{kijl} = 0.$$

Proof. We have

$$\begin{aligned} D_{\partial_j} D_{\partial_i} \partial_k =& D_{\partial_j}(\Gamma^m_{ik}\partial_m) \\ =& \partial_j \Gamma^m_{ik}\partial_m + \Gamma^m_{ik} D_{\partial_j}\partial_m \\ =& (\partial_j \Gamma^l_{ik} + \Gamma^m_{ik}\Gamma^l_{jm})\partial_l. \end{aligned}$$

Then the indentity holds since

$$R(\partial_i, \partial_j, \partial_k) = D_{\partial_j} D_{\partial_i} \partial_k - D_{\partial_i} D_{\partial_j} \partial_k.$$

Proposition. Let X be a smooth vector field along a smooth surface $f = f(s,t)$. Then

$$\partial_s \partial_t X - \partial_t \partial_s X = R(\partial_t f, \partial_s f)X.$$

Proof.

$$\begin{aligned} & \partial_t X \\ =& \partial_t (X^i \partial_i) = \partial_t X^i \partial_i + X^i \partial_t \partial_i \\ =& \partial_t X^i \partial_i + X^i D_{\partial_t f}\partial_i = \partial_t X^i \partial_i + X^i \partial_t f^j D_{\partial_j}\partial_i \\ \implies & \partial_s \partial_t X \\ =& \partial_s \partial_t X^i \partial_i + \partial_t X^i \partial_s f^j D_{\partial_j}\partial_i + \partial_s X^i \partial_t f^j D_{\partial_j}\partial_i \\ & + X^i \partial_s \partial_t f^j D_{\partial_j}\partial_i + X^i \partial_t f^j \partial_s f^k D_{\partial_k} D_{\partial_j}\partial_i \\ \implies & \partial_s \partial_t X - \partial_t \partial_s X \\ =& X^i \partial_t f^j \partial_s f^k D_{\partial_k} D_{\partial_j}\partial_i - X^i \partial_s f^j \partial_t f^k D_{\partial_k} D_{\partial_j}\partial_i \\ =& X^i \partial_t f^j \partial_s f^k [D_{\partial_k}, D_{\partial_j}]\partial_i \\ =& X^i \partial_t f^j \partial_s f^k R(\partial_j, \partial_k)\partial_i \\ =& R(\partial_t f^j \partial_j, \partial_s f^k \partial_k)(X^i \partial_i) \\ =& R(\partial_t f, \partial_s f)X. \end{aligned}$$

Sectional Curvature

Definition: sectional curvature. Define

$$\sec : \bigcup_{p\in\mathcal{M}} (T_p\mathcal{M})^2 \to \mathbb{R}, \quad \sec(u,v) = R(u,v,u,v), u, v \in T_p\mathcal{M}, p \in \mathcal{M}.$$

Then $\sec(u,u) = 0, \quad \sec(u,v) = \sec(v,u), \quad \sec(su,tv) = s^2t^2\sec(u,v).$

In this chapter, we often assume $n = \dim\mathcal{M} > 1$. Otherwise $R(\cdot,\cdot,\cdot,\cdot) = 0$.

Notation. Let $u, v \in T_p\mathcal{M}, u \not\parallel v$. Write

$$u \wedge v = \operatorname{span}\{u, v\}, \quad |u \wedge v| = \sqrt{|u|^2|v|^2 - \langle u, v\rangle^2}.$$

Definition: sectional curvature. Let

$$\Sigma = \{\sigma : \sigma \text{ is a 2-plane in } T_p\mathcal{M} \text{ for some } p \in \mathcal{M}\}.$$

Define

$$\sec : \Sigma \to \mathbb{R}, \quad \sec(u \wedge v) = \frac{\sec(u,v)}{|u \wedge v|^2}.$$

Then sec is well-defined.

Proof. Let $e_1 \wedge e_2 = u \wedge v$ with $e_1 \perp e_2, |e_1| = |e_2| = 1$ and

$$\begin{cases} u = ae_1 + be_2, \\ v = ce_1 + de_2. \end{cases}$$

Then

$$\begin{aligned} \sec(u,v) =& \sec(ae_1 + be_2, ce_1 + de_2) \\ =&(ad - bc)^2 \sec(e_1, e_2) \\ =&|u \wedge v|^2 \sec(e_1, e_2). \end{aligned}$$

The identity above, together with $|e_1 \wedge e_2| = 1$, implies that sec is well-defined.

Theorem.

$$R(x,y,u,v) = \frac{1}{6}\partial_s\partial_t\Big|_{(0,0)}\Big(\sec(x+su, y+tv) - \sec(x+sv, y+tu)\Big)$$

Proof. The identity holds since

$$\begin{aligned} &\partial_s\partial_t\Big|_{(0,0)} \sec(x+su, y+tv) \\ =&\partial_s\partial_t\Big|_{(0,0)} R(x+su, y+tv, x+su, y+tv) \\ =&R(x,y,u,v) + R(u,v,x,y) + R(x,v,u,y) + R(u,y,x,v) \\ =&2R(x,y,u,v) + 2R(x,v,u,y), \\ &\partial_s\partial_t\Big|_{(0,0)} \sec(x+sv, y+tu) \\ =&2R(x,y,v,u) + 2R(x,u,v,y) \\ =&-2R(x,y,u,v) - 2R(u,v,x,y) - 2R(v,x,u,y) \\ =&-4R(x,y,u,v) + 2R(x,v,u,y). \end{aligned}$$

Definition: manifold with constant sectional curvature. If $\sec(\sigma) = \kappa, \forall\sigma \in \Sigma$ for some $\kappa \in \mathbb{R}$, then we call $\mathcal{M}$ a manifold with constant sectional curvature κ.

Theorem. $\mathcal{M}$ is a manifolds with constant sectional curvature κ if and only if

$$R(u,v,x,y)=\kappa(\langle x,u\rangle\langle y,v\rangle-\langle x,v\rangle\langle u,y\rangle).$$

Proof. The "only if" part is obvious. We will only prove the "if" part. Indeed, by the previous theorem, we have

$$\begin{aligned}R(x,y,u,v)=&\frac{1}{6}\partial_s\partial_t\Big|_{(0,0)}\Big(\sec(x+su,y+tv)-\sec(x+sv,y+tu)\Big)\\=&\frac{\kappa}{6}\partial_s\partial_t\Big|_{(0,0)}\Big(|x+su|^2|y+tv|^2-\langle x+su,y+tv\rangle^2\\&\qquad\qquad-|x+sv|^2|y+tu|^2+\langle x+sv,y+tu\rangle^2\Big)\\=&\frac{\kappa}{6}\Big(4\langle x,u\rangle\langle y,v\rangle-2\langle x,y\rangle\langle u,v\rangle-2\langle x,v\rangle\langle u,y\rangle\\&\qquad-4\langle x,v\rangle\langle y,u\rangle+2\langle x,y\rangle\langle v,u\rangle+2\langle x,u\rangle\langle v,y\rangle\Big)\\=&\kappa(\langle x,u\rangle\langle y,v\rangle-\langle x,v\rangle\langle u,y\rangle).\end{aligned}$$

Spaces with Constant Sectional Curvature

Proposition. Let

$$f(z)=\frac{az+b}{cz+d},\quad z=x+iy\in\mathbb{C},$$

where $a,b,c,d\in\mathbb{R}, ad-bc=1$. Then
(i) $f(H^2)=H^2$.
(ii) $f(\mathbb{R}i)$ is a line $\perp$ x-axis or a semicircle centered at some point in x-axis.
(iii) $f:H^2\to H^2$ is a Riemannian isometry.

Proof. (i) For $w\in H^2$, choose $z=\frac{dw-b}{a-cw}$. Then $f(z)=w$ and

$$\begin{aligned}\mathrm{Im}z=&\frac{1}{2i}(z-\bar{z})=\frac{1}{2i}\left(\frac{dw-b}{a-cw}-\frac{d\bar{w}-b}{a-c\bar{w}}\right)\\=&\frac{(dw-b)(a-c\bar{w})-(d\bar{w}-b)(a-cw)}{2i(a-cw)(a-c\bar{w})}\\=&\frac{adw-ab-cd|w|^2+bc\bar{w}-ad\bar{w}+ab+cd|w|^2-bcw}{2i|a-cw|^2}\\=&\frac{w-\bar{w}}{2i|a-cw|^2}=\frac{\mathrm{Im}w}{|a-cw|^2}>0.\end{aligned}$$

(ii) By computation,

$$\mathrm{Re}z=\frac{(ad+bc)\mathrm{Re}w-cd|w|^2-ab}{|a-cw|^2}.$$

Let $\mathrm{Re}z=0$. If $cd=0$, then

$$\mathrm{Re}w=\frac{ab}{ad+bc}.$$

If $cd\neq 0$, then

$$\left(x-\frac{ad+bc}{2cd}\right)^2+y^2=\left(\frac{ad+bc}{2cd}\right)^2-\frac{ab}{cd}>0,$$

where $w=x+iy$.
(iii) From (i), we know that

$$\mathrm{Im}f(z)=\frac{\mathrm{Im}z}{|cz+d|^2}.$$

Thus for $z\in H^2, u,v\in T_zH^2=\mathbb{C}$, we have

$$\begin{aligned}
&df_z(u)=f'(z)u=\frac{u}{(cz+d)^2}\\
\Longrightarrow\quad&\langle df_z(u),df_z(v)\rangle_{H^2}\\
&=\frac{1}{(\mathrm{Im}f(z))^2}\mathrm{Re}\left(\frac{u}{(cz+d)^2}\frac{\bar{v}}{(c\bar{z}+d)^2}\right)\\
&=\frac{\mathrm{Re}(u\bar{v})}{(\mathrm{Im}f(z))^2|cz+d|^4}\\
&=\frac{\mathrm{Re}(u\bar{v})}{(\mathrm{Im}z)^2}=\langle u,v\rangle_{H^2}.
\end{aligned}$$

Theorem. Let

$$H^n=\{x=(x^1,\cdots,x^n)\in\mathbb{R}^n: x^n>0\}$$

be the hyperbolic space with $g_{ij}(x)=(x^n)^{-2}\delta_{ij}$. Then H^n has constant sectional curvature -1.

Proof. We have

$$\begin{aligned}
\Gamma_{ij}^k=&\frac{1}{2}(\partial_j g_{im}+\partial_i g_{jm}-\partial_m g_{ij})g^{mk}\\
=&(x^n)^{-1}(\delta_{mn}\delta_{ij}-\delta_{jn}\delta_{im}-\delta_{in}\delta_{jm})\delta_{mk}\\
=&(x^n)^{-1}(\delta_{kn}\delta_{ij}-\delta_{jn}\delta_{ik}-\delta_{in}\delta_{jk}).
\end{aligned}$$

Hence

$$\begin{aligned}
R_{ijk}^l=&\partial_j\Gamma_{ik}^l-\partial_i\Gamma_{jk}^l+\Gamma_{ik}^m\Gamma_{jm}^l-\Gamma_{jk}^m\Gamma_{im}^l\\
=&(x^n)^{-2}\Big(\delta_{in}(\delta_{ln}\delta_{jk}-\delta_{kn}\delta_{jl}-\delta_{jn}\delta_{kl})-\delta_{jn}(\delta_{ln}\delta_{ik}-\delta_{kn}\delta_{il}-\delta_{in}\delta_{kl})\\
&+(\delta_{mn}\delta_{ik}-\delta_{kn}\delta_{im}-\delta_{in}\delta_{km})(\delta_{ln}\delta_{jm}-\delta_{mn}\delta_{jl}-\delta_{jn}\delta_{ml})\\
&-(\delta_{mn}\delta_{jk}-\delta_{kn}\delta_{jm}-\delta_{jn}\delta_{km})(\delta_{ln}\delta_{im}-\delta_{mn}\delta_{il}-\delta_{in}\delta_{ml})\Big)\\
=&(x^n)^{-2}\Big(\delta_{in}(\delta_{ln}\delta_{jk}-\delta_{kn}\delta_{jl}-\delta_{jn}\delta_{kl})-\delta_{jn}(\delta_{ln}\delta_{ik}-\delta_{kn}\delta_{il}-\delta_{in}\delta_{kl})\\
&+\delta_{ik}(\delta_{ln}\delta_{jn}-\delta_{jl}-\delta_{jn}\delta_{ln})-\delta_{kn}(\delta_{ln}\delta_{ij}-\delta_{in}\delta_{jl}-\delta_{jn}\delta_{il})-\delta_{in}(\delta_{ln}\delta_{jk}-\delta_{kn}\delta_{jl}-\delta_{jn}\delta_{kl})\\
&-\delta_{jk}(\delta_{ln}\delta_{in}-\delta_{il}-\delta_{in}\delta_{ln})+\delta_{kn}(\delta_{ln}\delta_{ij}-\delta_{jn}\delta_{il}-\delta_{in}\delta_{jl})+\delta_{jn}(\delta_{ln}\delta_{ik}-\delta_{kn}\delta_{il}-\delta_{in}\delta_{kl}\Big)\\
=&(x^n)^{-2}(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jk}).
\end{aligned}$$

Thus

$$R_{ijkl}=R_{ijk}^m g_{ml}=(x^n)^{-4}(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jk}).$$

Write $e_i = |\partial_i|_{H^n}^{-1}\partial_i = x^n\partial_i$. For $\sigma\in\Sigma$, choose $u, v\in T_xH^n$ such that

$$\sigma = u\wedge v,\quad |u|_{H^n} = |v|_{H^n} = 1,\quad \langle u, v\rangle_{H^n} = 0,\quad u = u^ie_i,\quad v = v^ie_i.$$

Then $|u\wedge v| = 1$, and

$$\begin{aligned}\sec(\sigma) =&R(u, v, u, v) = u^iv^ju^kv^l(x^n)^4R_{ijkl}\\ =&u^iv^iv^ju^j - u^iu^iv^jv^j\\ =&\langle u, v\rangle_{H^n}^2 - |u|_{H^n}^2|v|_{H^n}^2 = -1.\end{aligned}$$

Therem. S^2 has constant sectional curvature 1.

Proof. Let

$$f(\theta,\varphi) = (\sin\theta, \cos\theta\sin\varphi, \cos\theta\cos\varphi),\quad \theta\in[0, 2\pi), \varphi\in[0,\pi].$$

It is sufficient to consider the curvature at $e_3 = f(0, 0)$. By computation,

$$\begin{cases}\partial_\theta f = (\cos\theta, -\sin\theta\sin\varphi, -\sin\theta\cos\varphi),\\ \partial_\varphi f = (0, \cos\theta\cos\varphi, -\cos\theta\sin\varphi)\end{cases}\implies\begin{cases}\partial_\theta f(0,0) = e_1,\\ \partial_\varphi f(0,0) = e_2.\end{cases}$$

Let $X = \partial_\theta f$. Then

$$\begin{gathered}\partial_\theta X = T_f(-\sin\theta, -\cos\theta\sin\varphi, -\cos\theta\cos\varphi) = 0,\\ \partial_\varphi\partial_\theta X = 0,\\ \partial_\varphi X = T_f(0, -\sin\theta\cos\varphi, \sin\theta\sin\varphi) = (0, -\sin\theta\cos\varphi, \sin\theta\sin\varphi),\\ \partial_\theta\partial_\varphi X = T_f(0, -\cos\theta\cos\varphi, \cos\theta\sin\varphi) = (0, -\cos\theta\cos\varphi, \cos\theta\sin\varphi).\end{gathered}$$

Thus

$$\partial_\varphi\partial_\theta X - \partial_\theta\partial_\varphi X = R(\partial_\theta f, \partial_\varphi f)X.$$

Choose $\theta = \varphi = 0$, and we get

$$e_2 = R(e_1, e_2)e_1 \implies R(e_1, e_2, e_1, e_2) = 1.$$

Since $T_{e_3}S^2$ is a linear space with dimension 2, S^2 has constant sectional curvature 1 at e_3.

Theorem. S^n has constant sectional curvature.

Proof. In the proof, we will use same notation of e_i in both $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. And we only compuate the curvature at e_{n+1}. Let

$$f(x) = (x, \sqrt{1-|x|^2}),\quad x\in\mathbb{R}^n, |x| < 1.$$

Then

$$\begin{aligned}\partial_if =&\Big(e_i, -x_i(1-|x|^2)^{-\frac{1}{2}}\Big),\quad \partial_if(0) = e_i,\\ \partial_j\partial_if =&T_f\Big(0, -\delta_{ij}(1-|x|^2)^{-\frac{1}{2}} - x_ix_j(1-|x|^2)^{-\frac{3}{2}}\Big)\\ =&\Big((\delta_{ij} + \frac{x_ix_j}{1-|x|^2})x, -|x|^2(\delta_{ij}(1-|x|^2)^{-\frac{1}{2}} + x_ix_j(1-|x|^2)^{-\frac{3}{2}})\Big),\\ \partial_k\partial_j\partial_if(0) =&\delta_{ij}e_k.\end{aligned}$$

Hence

$$\begin{aligned}
&\partial_k\partial_j\partial_i f - \partial_j\partial_k\partial_i f = R(\partial_j f, \partial_k f)\partial_i f \\
\implies &\delta_{ij}e_k - \delta_{ik}e_j = R(e_j, e_k)e_i \\
\implies &R(e_j, e_k, e_i, e_l) = \delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}.
\end{aligned}$$

For $\sigma = u \wedge v \in \Sigma$ with $|u| = |v| = 1, u \perp v$, we have

$$\begin{aligned}
\sec(\sigma) =&R(u, v, u, v) = u_j v_k u_i v_l R(e_j, e_k, e_i, e_l) \\
=&u_i^2 v_k^2 - u_i v_i u_j v_j = |u|^2|v|^2 - \langle u, v\rangle^2 = 1.
\end{aligned}$$

Ricci Curvature and Scalar Curvature

Trace of bilinear operator. Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space with dimension n and B a bilinear operator on V. Then there exists a unique linear operator $A : V \to V$ such that

$$B(u, v) = \langle Au, v\rangle, \quad u, v \in V.$$

Define $\operatorname{tr} B := \operatorname{tr} A$.
Let $e_1, \cdots, e_n$ be a basis of V. Write

$$g_{ij} = \langle e_i, e_j\rangle, \quad (g^{ij}) = (g_{ij})^{-1}, \quad Ae_i = A_i^j e_j, \quad B_{ij} = B(e_i, e_j).$$

Then we have

$$\begin{aligned}
B_{ij} &= \langle Ae_i, e_j\rangle = A_i^k g_{kj} \\
\implies A_i^j &= B_{ik}g^{kj}, \quad \operatorname{tr} B = \operatorname{tr} A = A_i^i = B_{ij}g^{ji}.
\end{aligned}$$

Given a tenser $T : \mathcal{V}(\mathcal{M}) \times \mathcal{V}(\mathcal{M}) \to C^\infty(\mathcal{M})$, define $\operatorname{tr} T : \mathcal{M} \to \mathbb{R}$ by

$$\operatorname{tr} T(p) = \text{ trace of } T : T_p\mathcal{M} \times \mathcal{M} \to \mathbb{R}.$$

Then

$$\operatorname{tr} T = T_{ij}g^{ij} \in C^\infty(\mathcal{M}),$$

where $T_{ij} = T(\partial_i, \partial_j)$.

Let $T : \mathcal{V}^k(\mathcal{M}) \to C^\infty(\mathcal{M})$ be a k-tensor with $k \geq 3$. Define

$$T_{(\alpha,\beta)}(x_1, \cdots, \hat{x}_\alpha, \cdots, \hat{x}_\beta, \cdots, x_k) = \operatorname{tr} T(x_1, \cdots, x_{\alpha-1}, \cdot, x_{\alpha+1}, \cdots, x_{\beta-1}, \cdot, x_{\beta+1}, \cdots, x_k).$$

Then $T_{(\alpha,\beta)}$ is a $(k-2)$-tensor and

$$(T_{(\alpha,\beta)})_{i_1,\cdots,\hat{i}_\alpha,\cdots,\hat{i}_\beta,\cdots,i_k} = T_{i_1,\cdots,i_k}g^{i_\alpha i_\beta}.$$

$R(\cdot,\cdot,\cdot,\cdot)$ is a tensor of order 4. We have

$$R_{(1,2)} = R_{(3,4)} = R_{ijkl}g^{ij} = 0, \quad R_{(2,4)} = R_{(1,3)} = -R_{(1,4)} = -R_{(2,3)}.$$

Definition: Ricci curvature. Define $\operatorname{Ric} = R_{(2,4)}$. Then $(\operatorname{Ric})_{ij} = R_{ikjl}g^{kl}$.

Let $T_p\mathcal{M} = \operatorname{span}\{e_i\}_{i=1}^n$ with $\langle e_i, e_j\rangle = \delta_{ij}$. Then

$$\operatorname{Ric}(e_i, e_j) = \operatorname{tr} R(e_i, \cdot, e_j, \cdot) = R(e_i, e_k, e_j, e_l)\delta_{kl} = R(e_i, e_k, e_j, e_k).$$

When $n=2$,

$$\mathrm{Ric}(e_1,e_1)=R(e_1,e_2,e_1,e_2)=\sec(e_1,e_2).$$

When $n=3$,

$$\mathrm{Ric}(e_i,e_i)=\sec(e_i,e_j)+\sec(e_i,e_k),\quad \{i,j,k\}=\{1,2,3\}.$$

Thus Ric and $\sec$ are equivalent when $n\le 3$.

Definition: scalar curvature. Define $\mathrm{scal}=\mathrm{tr}(\mathrm{Ric})=(\mathrm{Ric})_{ij}g^{ij}=R_{ikjl}g^{ij}g^{kl}$.

If $\mathcal{M}$ has constant sectional curvature κ, then

$$\mathrm{scal}=\mathrm{Ric}(e_i,e_j)\delta_{ij}=\sec(e_i,e_k)=n(n-1)\kappa.$$

Jacobi Field

Introduction

Definition: Jacobi field. Let $\gamma:[0,a]\to\mathcal{M}$ be a geodesic, $J\in\mathcal{V}(\gamma)$. If J satisfies the Jacobi equation:

$$J''+R(\gamma',J)\gamma'=0,$$

we call J a Jacobi field along γ.

Proposition. Let $\gamma:[0,a]\to\mathcal{M}$ be a geodesic, $J\in\mathcal{V}(\gamma)$. Then J is a Jacobi field if and only if $\exists$ a geodesic variation $f=f(s,t):(-\varepsilon,\varepsilon)\times[0,a]\to\mathcal{M}$ (i.e. $f_0=\gamma,\partial_t^2 f=0,\forall s$) such that $J=\partial_s f\big|_{s=0}$.

Proof. The "if" part. By computation,

$$\begin{aligned}0=&\partial_s\partial_t^2 f\big|_{s=0}=\partial_t\partial_s\partial_t f\big|_{s=0}+R(\partial_t f,\partial_s f)\partial_t f\big|_{s=0}\\=&\partial_t^2\partial_s f\big|_{s=0}+R(\gamma',J)\gamma'=J''+R(\gamma',J)\gamma'.\end{aligned}$$

The "only if" part. Choose $\alpha:(-\varepsilon,\varepsilon)\to\mathcal{M}$ and $X\in\mathcal{V}(\alpha)$ such that

$$\alpha(0)=\gamma(0),\quad \alpha'(0)=J(0),\quad X(0)=\gamma'(0),\quad X'(0)=J'(0).$$

Let

$$f(s,t)=\exp_{\alpha(s)}(tX(s)),\quad K(t)=\partial_s f(0,t).$$

Then

$$\begin{aligned}K(0)=\partial_s\big|_{s=0}f(s,0)=\partial_s\big|_{s=0}\alpha(s)=\alpha'(0)=J(0),\\K'(0)=\partial_t\partial_s f(0,0)=\partial_s\partial_t f(0,0)=X'(0)=J'(0).\end{aligned}$$

Thus $J=K$.

For $\gamma(0)=p,v,w\in T_p\mathcal{M}$, consider the ODE

$$\begin{cases}J''+R(\gamma',J)\gamma'=0,\\J(0)=v,J'(0)=w.\end{cases}$$

Let $\mathscr{J}$ be the set of all Jacobi fields with initial value in $T_p\mathcal{M} \times T_p\mathcal{M}$ and write

$$S : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathscr{J}, \quad S(v, w) = J.$$

Then S is a isomorphism and $\dim \mathscr{J} = 2n$. We also write

$$\mathscr{J}^\perp = \{J \in \mathscr{J} : J(t) \perp \gamma'(t), \forall t \in [0, a]\}.$$

Proposition. Let $J \in \mathscr{J}$. Then $J \in \mathscr{J}^\perp \iff J(0), J'(0) \perp \gamma'(0)$.

Proof.

$$\begin{aligned}
&\partial_t^2 \langle J, \gamma' \rangle = \langle J'', \gamma' \rangle = -R(\gamma', J, \gamma', \gamma') = 0 \\
\implies &\langle J, \gamma' \rangle = \langle J'(0), \gamma'(0) \rangle t + \langle J(0), \gamma'(0) \rangle.
\end{aligned}$$

Proposition. $\mathscr{J} = \mathscr{J}^\perp \oplus \operatorname{span}\{\gamma', t\gamma'\}$.

Proof. Observe that $\gamma' \not\parallel t\gamma', \gamma', t\gamma' \perp \mathscr{J}^\perp$ and

$$\dim \mathscr{J}^\perp = \dim\{J \in \mathscr{J} : J(0), J'(0) \perp \gamma'(0)\} = 2n - 2.$$

Theorem. Let $\gamma : [0, a] \to \mathcal{M}$ be a geodesic with $\gamma(0) = p, \gamma'(0) = v, |v| = 1$, J a Jacobi field along γ with $J(0) = 0, J'(0) = w, |w| = 1, w \perp v$. Then

$$|J(t)| = t - \frac{1}{6} \sec(v, w) t^3 + o(t^3) \quad \text{as } t \to 0.$$

Proof. By computation,

$$\begin{aligned}
\partial_t\big|_{t=0} |J|^2 =& 2\langle J'(0), J(0) \rangle = 0, \\
\partial_t^2\big|_{t=0} |J|^2 =& 2\langle J''(0), J(0) \rangle + 2|J'(0)|^2 = 2, \\
\partial_t^3\big|_{t=0} |J|^2 =& 2\langle J'''(0), J(0) \rangle + 6\langle J''(0), J'(0) \rangle = -6R(\gamma'(0), J(0), \gamma'(0), J'(0)) = 0, \\
\partial_t^4\big|_{t=0} |J|^2 =& 2\langle J''''(0), J(0) \rangle + 8\langle J'''(0), J'(0) \rangle + 6|J''(0)|^2 = 8\langle J'''(0), J'(0) \rangle \\
=& -8\langle \partial_t\big|_{t=0} R(\gamma', J, \gamma'), w \rangle = -8\langle \partial_t\big|_{t=0} \big(J^i R(\gamma', \partial_i, \gamma')\big), w \rangle \\
=& -8\langle (J^i)' R(\gamma', \partial_i, \gamma')\big|_{t=0}, w \rangle = -8\langle R(\gamma', J', \gamma')\big|_{t=0}, w \rangle \\
=& -8R(v, w, v, w) = -8\sec(v, w).
\end{aligned}$$

Hence

$$|J(t)|^2 = t - \frac{1}{3} \sec(v, w) t^4 + o(t^4) = \left(t - \frac{1}{6} \sec(v, w) t^3 + o(t^3)\right)^2 \quad \text{as } t \to 0.$$

Conjugate Points

Definition: conjugate points. Let $\gamma : [0, a] \to \mathcal{M}$ be a geodesic with $p = \gamma(0), q = \gamma(t_0)$ for some $t_0 \in (0, a]$. If $\exists\, J \in \mathscr{J} \setminus \{0\}$ with $J(0) = 0, J(t_0) = 0$, we say q is a conjugate point of p.

Example. In S^2, $-p$ is a conjugate point of $p \in S^2$.

Proof. It is sufficient to consider the case when $p = e_3$. Let

$$f(s, t) = (\sin s \sin t, \cos s \sin t, \cos t), \quad \gamma = f_0.$$

Then $e_3 = \gamma(0), -e_3 = \gamma(\pi)$ and f is a geodesic variation. Let $J = \partial_s\big|_{s=0} f$ be a Jacobi field. Then

$$J(t) = (\sin t, 0, 0) \implies J(0) = 0, J(\pi) = 0.$$

Let

$$\begin{aligned}\mathscr{J}_0 &= \{J \in \mathscr{J} : J(0) = 0\} = S(\{0\} \times T_p\mathcal{M}),\\ \mathscr{J}_{0,0} &= \{J \in \mathscr{J} : J(0) = 0, J(t_0) = 0\}.\end{aligned}$$

Then q is a conjugate point of $p \iff \dim \mathscr{J}_{0,0} > 0$. Thus we define

$$\operatorname{mul} q = \dim \mathscr{J}_{0,0} < n,$$

since $t\gamma' \in \mathscr{J}_0 \setminus \mathscr{J}_{0,0}$.

Definition: critical point. Let $f \in C^\infty(\mathcal{M}_1; \mathcal{M}_2), m_1 \in \mathcal{M}_1$. If df_{m_1} is not surjective, then we call q a critical point of f.
When $\dim \mathcal{M}_1 = \dim \mathcal{M}_2$, q is a critical point of $f \iff df_{m_1}$ is not injective $\iff \dim\ker(df_{m_1}) > 0$.

Since γ is a geodesic, $\gamma(t) = \exp_p(t\gamma'(0))$ and $q = \exp_p(v_0)$, where $v_0 = t_0\gamma'(0)$.

Proposition. q is a conjugate point of $p \iff v_0$ is a critical point of $\exp_p$. In this case, $\operatorname{mul} q = \dim\ker d(\exp_p)_{v_0}$.

Proof. We claim that if $J \in \mathscr{J}_0$,

$$J(t) = \partial_s\big|_{s=0} \exp_p t(\gamma'(0) + sJ'(0)) = td(\exp_p)_{t\gamma'(0)}(J'(0)).$$

Let $f(s,t) = \exp_p t(\gamma'(0) + sJ'(0))$. Then $f_0 = \gamma, f_s'' = 0, \forall s$. Moreover,

$$\begin{aligned}\partial_s\big|_{s=0} \exp_p t(\gamma'(0) + sJ'(0))\big|_{t=0} &= 0 = J(0),\\ \partial_t\partial_s\big|_{s=0} \exp_p t(\gamma'(0) + sJ'(0))\big|_{t=0} &= \partial_s\big|_{s=0}\partial_t\big|_{s=0} \exp_p t(\gamma'(0) + sJ'(0))\\ &= \partial_s\big|_{s=0}(\gamma'(0) + sJ'(0)) = J'(0).\end{aligned}$$

Hence we prove the claim. Then

$$J \in \mathscr{J}_{0,0} \iff J'(0) \in \ker d(\exp_p)_{v_0}.$$

Thus

$$\mathscr{J}_{0,0} = S(\{0\} \times \ker d(\exp_p)_{v_0}).$$

Then the proof is finished.

Jacobi Fields on a Manifold with Constant Sectional Curvature

Definition: parallel vector field. Let $\alpha : [a,b] \to \mathcal{M}$ be a smooth curve, $X \in \mathcal{V}(\alpha)$. If

$$\partial_t X \equiv 0,$$

we say X is a parallel vector field along α. Set

$$\mathscr{X} = \{X \in \mathcal{V}(\alpha) : \partial_t X \equiv 0\}.$$

Then for $X, Y \in \mathscr{X}$, $\langle X, Y\rangle \equiv \text{constant}$.

Definition: geodesic frame. Let $\gamma : [0, a] \to \mathcal{M}$ is a geodesic with $\gamma(0) = p$. Let $T_p\mathcal{M} = \text{span}\{e_i\}_{i=1}^n$ with $\langle e_i, e_j\rangle = \delta_{ij}$. And let E_i be the parallel vector field along γ with $E_i(0) = e_i$. Then

$$\langle E_i(t), E_j(t)\rangle = \delta_{ij}, \quad T_{\gamma(t)}\mathcal{M} = \text{span}\{E_i(t)\}_{i=1}^n, \quad \forall t \in [0, a].$$

Then $\{E_i\}_{i=1}^n$ is called a geodesic frame.

Let $\mathcal{M}$ be a Riemannian manifold with constant sectional curvature κ, $\gamma : [0, a] \to \mathcal{M}$ be a geodesic with $\gamma(0) = p, |\gamma'| = 1$. Then

$$\begin{aligned} & R(X, Y, Z, W) = \kappa(\langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle) \\ \Longrightarrow\ & R(X, Y)Z = \kappa(\langle X, Z\rangle Y - \langle Y, Z\rangle X). \end{aligned}$$

Choose a geodesic frame $\{E_i\}_{i=1}^n$ along γ such that $E_n(0) = \gamma'(0)$. Then $E_n = \gamma'$. For $J \in \mathscr{J}$, write $J = \alpha_i E_i$. Then

$$\begin{aligned} J'' &= \alpha_i'' E_i, \\ R(\gamma', J)\gamma' &= \kappa(\langle\gamma', \gamma'\rangle J - \langle J, \gamma'\rangle\gamma') = \kappa\sum_{i=1}^{n-1} \alpha_i E_i. \end{aligned}$$

Hence $J'' + R(\gamma', J)\gamma' = 0$ implies

$$\alpha_i'' + \kappa\alpha_i = 0, \quad i = 1, \cdots, n-1, \quad \alpha_n'' = 0.$$

Then

$$J(t) = \begin{cases} \sum\limits_{i=1}^{n-1} \left(a_i \frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}} + b_i \cos(\sqrt{\kappa}t)\right) E_i + (a_n t + b_n)E_n, & \kappa > 0, \\ \sum\limits_{i=1}^{n} (a_i t + b_i)E_i, & \kappa = 0, \\ \sum\limits_{i=1}^{n-1} \left(a_i \frac{\sinh(\sqrt{\kappa}t)}{\sqrt{\kappa}} + b_i \cosh(\sqrt{\kappa}t)\right) E_i + (a_n t + b_n)E_n, & \kappa < 0. \end{cases}$$

When $\mathcal{M} = S^n$, $\gamma : [0, 2\pi] \to \mathcal{M}$ with $p = \gamma(0), |\gamma'| = 1, q = \gamma(t_0)$, we have

$$J(t) = \sum_{i=1}^{n-1} (a_i \sin t + b_i \cos t)E_i + (a_n t + b_n)\gamma' \in \mathscr{J}.$$

Then $J \in \mathscr{J}_{0,0} \setminus \{0\} \iff$

$$b_1 = \cdots = b_n = a_n = 0, \quad t_0 = \pi.$$

Hence q is a conjugate point of p if and only if $q = \gamma(\pi) = -p$. In this case, $\text{mul}\, q = n - 1$.